LATTICE ISOMORPHISM BETWEEN SPACES OF INTEGRABLE FUNCTIONS WITH RESPECT TO POSITIVE VECTOR MEASURE

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Abstract: This paper is concerned with lattice isomorphisms between spaces of integrable functions with respect to positive vector measure to study the relation between different spaces of vector measures \((\Omega_1,\Sigma_1,m_1)\) and \((\Omega_2,\Sigma_2,m_2)\) where \((\Omega_1,\Sigma_1)\) and \((\Omega_2,\Sigma_2)\) are measurable spaces and \(m_1\) and \(m_2\) are countably additive vector measures taking values in real Banach spaces \(X\) and \(Y\) respectively, when the corresponding spaces of integrable functions \(L^1(m_1)\) and \(L^1(m_2)\) are lattice isomorphic.

Keywords: Banach space, Normed linear space, Lattice isomorphism, Vector measure, Vector Lattice.

1. INTRODUCTION

At the beginning of this century the works of Lebesgue and other mathematicians created a modern and complete theory of integration which allowed to integrate in a fully satisfactory way a broad class of real functions with respect to a positive measure. Among the several directions of development of this theory it was the work of Bochner in 1933 who created a Lebesgue type theory in order to integrate vector valued functions with respect to a positive measure.

The technique that they use for this study is to represent the action of the operator as integration with respect to a measure, associated to the operator, with values in the Banach space \(X\). In this way they create a theory for integrating scalar functions with respect to a measure defined on a \(\mathcal{A}\)-algebra and with values in a Banach space.

This paper is organized as follows. Section 2 deals with the definitions of lattice isomorphisms with respect to positive vector measure and Section 3 deals with the lattice isomorphism theorems by using integrable functions and Section 4 concludes the paper.

2. PRELIMINARIES

2.1 Lattice

A lattice is a partially ordered set in which any two elements have a unique supremum (also called a least upper bound or join) and a unique infimum (also called a greatest lower bound or meet).

2.2 Partially ordered set

A partial order is a binary relation \(\leq\) over a set \(P\) which is reflexive, antisymmetric, and transitive, i.e., for all \(a, b,\) and \(c\) in \(P,\) we have that:

- \(a \leq a\) (reflexivity);
- if \(a \leq b\) and \(b \leq a\) then \(a = b\) (antisymmetry);
- if \(a \leq b\) and \(b \leq c\) then \(a \leq c\) (transitivity).

2.3 Lattice Isomorphism

Let \(L = (L,\wedge,\vee)\) and \(K = (K,\wedge,\vee)\) be lattices and let \(h : L \rightarrow K.\) A lattice isomorphism is a one-to-one and onto lattice homomorphism.

2.4 Lattice homomorphism

Let \(L = (L,\wedge,\vee)\) and \(K = (K,\wedge,\vee)\) be lattices and let \(h : L \rightarrow K\) then \(h\) is a lattice homomorphism of any \(a, b \in L, h(a\vee b) = h(a) \vee h(b)\) & \((a\wedge b) = h(a) \wedge h(b).\) In other words, the mapping \(h\) is a lattice homomorphism if it is both join-homomorphism and meet a homomorphism.
2.5 \(L^p\) Spaces

Let \(1 \leq p < \infty\) and \((S, \Sigma, \mu)\) be a measure space. Consider the set of all measurable functions from \(S\) to \(C\) (or \(R\)) whose absolute value raised to the \(p\)-th power has finite integral, or equivalently, that

\[
\|f\|_p = \left(\int_S |f|^p \, d\mu\right)^{\frac{1}{p}} < \infty
\]

such functions forms a vector space, with the following natural operations:

\[
(f + g)(x) = f(x) + g(x), \quad \text{and} \quad (\lambda f)(x) = \lambda f(x)
\]

2.6 Vector Measure

Given a field of sets \((\Omega, \mathcal{F})\) and a Banach space \(X\), a finitely additive vector measure (or measure, for short) is a function \(\mu : \mathcal{F} \to X\) such that for any two disjoint sets \(A\) and \(B\) in \(\mathcal{F}\) one has

\[
\mu(A \cup B) = \mu(A) + \mu(B).
\]

called countably additive if for any sequence \(A_i\) of disjoint sets in \(\mathcal{F}\) such that their union is in \(\mathcal{F}\) it holds that

\[
\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)
\]

with the series on the right-hand side convergent in the norm of the Banach space \(X\).

2.7 Integrable

A measurable function \(f : \Omega \to R\) is called integrable with respect to \(m\) if:

(i) \(f \in L^1(|m(x)|)\) for all \(x' \in X'\), and

(ii) for each \(A \in \Sigma\) there exists an element \(f dm \in X\)

\(A\)

(called the integral of \(f\) over \(A\)) such that

\[
\langle f dm, x' \rangle = \int_A f dm(x') \text{ for all } x' \in X'
\]

3. Lattice isomorphisms with respect to positive vector measure

Result 3.1 :

A lattice isomorphism between \(L^1(m_1)\) and \(L^1(m_2)\) coming from a Boolean Algebra isomorphism \(\Phi : \Sigma_1 \to \Sigma_2\) we need to impose some more extra conditions on \(\Phi\).

Namely

(C1) There is a constant \(K_1 > 0\) such that for each \(0 \leq y' \in B Y'\)

and each \(\pi \in \Pi(\Omega_1)\) there exists \(0 \leq x' \in B X'\) satisfying that

\[
\langle m_1(\Phi(A)), y' \rangle \leq K_1 \langle m_1(A), x' \rangle, A \in \pi.
\]

(C2) There is a constant \(K_2 > 0\) such that for each \(0 \leq x' \in B X'\) and each \(\pi \in \Pi(\Omega_1)\) there exists \(0 \leq y' \in B Y'\) satisfying that

\[
\langle m_2(\Phi(A)), y' \rangle \leq K_2 \langle m_2(A), x' \rangle, A \in \pi.
\]
Result 3.2

If $\Phi : \Sigma_1 [m_1] \rightarrow \Sigma_2 [m_2]$ is an isomorphism of Boolean algebras then there is unique isometric multiplicative lattice isomorphism

$$T : L^\infty(m_1) \rightarrow L^\infty(m_2)$$
such that

$$T(\chi_A) = \chi_{\Phi(A)} \text{ for all } A \in \Sigma_1.$$ 

In particular $T(\chi_{\Omega_1}) = \chi_{\Omega_2}.$

Result 3.3

If $T : L^\infty(m_1) \rightarrow L^\infty(m_2)$ is a lattice isomorphism, then

$$\Phi : [A] \in \Sigma[m_1] \rightarrow \Phi(A) := [\text{supp}(T\chi_A)] \in \Sigma[m_2]$$
is an isomorphism of Boolean algebras such that $T(\chi_A) = T(\chi_{\Phi(A)}) \text{ for all } A \in \Sigma_1.$ If moreover $T$ is multiplicative then $T(\chi_A) = \chi_{\Phi(A)} \text{ for all } A \in \Sigma_1.$

Theorem 3.4

Let $\Phi : \Sigma_1 [m_1] \rightarrow \Sigma_2 [m_2]$ be an isomorphism of Boolean algebras such that the conditions (C1) and (C2) hold. Then there exists a unique lattice isomorphism

$$T : L^1(m_1) \rightarrow L^1(m_2)$$
such that $T(L^\infty(m_1)) \subseteq L^\infty(m_2).$ Moreover, the restriction

$$T : L^\infty(m_1) \rightarrow L^\infty(m_2)$$
is an isometric multiplicative lattice isomorphism satisfying

$T(\chi_A) = \chi_{\Phi(A)} \text{ for all } A \in \Sigma_1.$ In particular, $T(\chi_{\Omega_1}) = \chi_{\Omega_2}.$

Proof:

Suppose that $\Phi : \Sigma_1 [m_1] \rightarrow \Sigma_2 [m_2]$ is an isomorphism of Boolean algebras. For each simple function

$$N \phi = \sum_{k=1}^{N} a_k \chi_{A_k} \text{ of } L^1(m_1) \text{ we define}$$

$$T(\phi) := \sum_{k=1}^{N} a_k \chi_{\Phi(A_k)}$$

which is a simple function in $L^1(m_2).$ As in the $L^\infty$ case we have defined a map

$$T : S(m_1) \rightarrow S(m_2)$$
from the set of simple functions of $L^1(m_1)$ into the set of simple functions of $L^1(m_2)$ that is a multiplicative and lattice preserving linear bijection. Moreover, it satisfies

$$||T(\phi)||_{L^\infty(m_2)} = ||\phi||_{L^\infty(m_1)}$$
for all $\phi \in S(m_1).$

Let us prove now that the requirements (C1) and (C2) in the statement of the theorem imply continuity of the operator $T : S(m_1) \rightarrow S(m_2)$ and its inverse with respect to the norms $||.||_{L^1(m_1)}$ and $||.||_{L^1(m_2)},$ respectively. Since the measure $m_2$ is positive, for each simple function

$$N \phi = \sum_{k=1}^{N} a_k \chi_{A_k}$$
an application of Hahn Banach theorem gives

$$||T(\phi)||_{L^1(m_2)} = \int_{\Omega_2} |T(\phi)(m_2)| dy$$

for $N \phi.$
\[ \sum_{k=1}^{N} 1 \{ a_k l m_2(\varphi(A_k)) \} \]

\[ \langle \sum_{k=1}^{N} a_k l m_2(\varphi(A_k)), y' \rangle \]

\[ = \sum_{k=1}^{N} 1 \{ a_k l m_2(\varphi(A_k)) \} \langle m_2(\varphi(A_k)), y' \rangle \]

For a certain \( 0 \leq y' \in B_{Y} \) from (C1) we obtain there exists \( 0 \leq x' \in B_{X} \) such that

\[ \langle m_2(\Phi(A_k)) y' \rangle \leq K_1 \langle m_1(A_k), x' \rangle \]

for every \( k = 1, 2, \ldots, N \). Thus (3.1) implies

\[ \| T(\varphi) \| _{L^1(m_2)} = \sum_{k=1}^{N} |a_k| \langle m_2(\varphi(A_k)), y' \rangle \]

\[ \leq K_1 \sum_{k=1}^{N} |a_k| \langle m_1(A_k), x' \rangle \]

\[ \leq K_1 \sum_{k=1}^{N} |a_k| \| m_1(A_k) \| _{X} \]

\[ \leq K_1 \| \varphi \| _{L^1(m_1)} \]

A similar argument proves that \( \| \varphi \| _{L^1(m_1)} \leq k_2 \| T(\varphi) \| _{L^1(m_2)} \) for every \( \varphi \in S(m_1) \). Therefore can be expanded from \( L^1(m_1) \) to \( L^1(m_2) \) and the extension satisfied the conditions in the statement of the theorem, since \( S(m_1) \) is dense in \( L^1(m_2) \).

The uniqueness and properties of \( T: L^\infty(m_1) \to L^\infty(m_2) \) are clearly seen for result 3.2.

On the opposite way, we have the one that follows, that is the analogous result to result 3.3.

**Conclusion**

In this paper, lattice isomorphisms with respect to positive vector measure are investigated with the help of integrable functions.

**REFERENCES**