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Best Generalized inverse of Linear Operator in Normed affine space

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***Abstract*: Let X,Y be normed affine space $A(X,Y)$ be a bounded linear operator from X to Y one wants to solve the linear problem $Ax = y$ for x (given $y \in Y$). When A is invertible, the unique solution is $x = A^{-1}y$. If this is not the case, one seeks an approximate solution of the form $x = By$, where B is an operator from Y to X such B is called generalized inverse of A. Given an affine space E of dim n and an affine frame $(a_0, a_1 \cdots a_m)$ for E. Let $f: E \to E$ and $g: E \to E$ be two affine maps represented by the two $(n+1) \times (n+1)$ matrices $\begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} B & c \\ 0 & 1 \end{bmatrix}$ with respect to the frame $(a_0, a_1 \cdots a_m)$ we also say that f and g represent by (A, b) and (b, c). In this paper we prove that f is invertible iff A is invertible and the solution exists in a unique way.**

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I.INTRODUCTION

Affine maps is an rigid motion but not a linear map in general. Also given an $m \times n$ matrix A and a vector $b \in \mathbb{R}^m$ the set $u = \{x \in R^n / Ax = b\}$ of solution of the system $Ax = b$ is an affine space but not a vector space in general. Here we take advantage of the fact that almost every affine concept is the counterpart of some concept in linear algebra. We begin by defining affine space corresponding to linear combinations of vector, define affine combination of point and linear subspace, introduce affine subspaces as subset closed under affine combinations corresponding to linear independence and bases we define affine independence and affine frames, affine maps. At the end corresponding to linear functional with $(n+1) \times (n+1)$ matrices by $(f \circ g)$ exists a invertible unique solution, there corresponds a unique invertible solution in the affine space.

II.PRELIMINARIES

In this section, we list some basic definitions.

Definition 2.1 *An affine space is either the degenerate space reduced to the empty set, or a triple $< E, \vec{E}, + >$ consisting of a non-empty set E (of points), a vector space $\vec{E}$ (of translations, or free vectors), and an action $+: E \times \vec{E} \to E$ satisfying the following conditions.*

(i) $a + 0 = a$, for every $a \in E$.

(ii) $(a + u) + v = a + (u + v), \forall a, b \in E$, and $u, v \in \vec{E}$.

(iii) For any two points $a, b \in E$, there is a unique $u \in \vec{E}$ such that $a + u = b$. The unique vector $u \in \vec{E}$ such that $a + u = b$ is denoted by $\overrightarrow{ab}$, or sometimes by ab, or even by $b - a$.

(ie) $b = a + \overrightarrow{ab}$ (or) $b = a + \boldsymbol{ab}$ (or) $b = a + (b - a)$.

Definition 2.2 1*The dimension of the affine space $< E, \vec{E}, + >$ is the dimension $dim(\vec{E})$ of the vector space $\vec{E}$. The conditions (i) and (ii) says that the(abelian) group $\vec{E}$ acts on E, and (iii) says that $\vec{E}$ acts transitively and faithfully on E.*

Definition 2.3 2 *A fundamental concept in linear algebra is that of a linear combination[1]. The corresponding concept in affine geometry is that of an affine combination called barycenter.*

Definition 2.4 3 *For any family of points $(a_i)_{i\in I}$ in E, for any family $(\lambda_i)_{i\in I}$ of scalars such that $\sum_{i \in I} \lambda_i$ and for any $a \in E$, the point $a + \sum_{i\in I} (\lambda_i)\overrightarrow{aa}_i$ [Which is independent of $a \in E$], is called barycenter of the points a_i, assigned the weights λ_i and denoted by $\sum_{i\in I} \lambda_i a_i$.*

Definition 2.5 *Given an affine space $< E, \vec{E}, + >$, a subset V of E is an affine subspace of $< E, \vec{E}, + >$ if for every family of weighted points $((a_i, \lambda_i))_{i\in I}$ in V such that $\sum_{i\in I} \lambda_i = 1$, the barycenter $\sum_{i\in I} \lambda_i a_i$ belongs to V.*

(*) An affine subspace is also called a flat.

(*) The empty set is trivially an affine subspace.

(*) Every intersection of affine subspace is an affine subspace.

Definition 2.6 4 *Given an affine space* $< E, \vec{E}, +>$, *an affine frame with origin* a_0 *is a family* $(a_0, a_1 \cdots a_m)$ *of* $m+1$ *points in* E *the list of vectors* $(\overrightarrow{a_0 a_1}, \overrightarrow{a_0 a_2}, \cdots \overrightarrow{a_0 a_m})$ *is a basis of* $\vec{E}$. *The pair* $(a_0, (\overrightarrow{a_0 a_1}, \overrightarrow{a_0 a_2} \cdots \overrightarrow{a_0 a_m}))$ *is also called an affine frame with origin* a_0. *Then every* $x \in E$ *can be expressed as* $x = a_0 + x_1 \overrightarrow{a_0 a_1} + x_2 \overrightarrow{a_0 a_2} \cdots + x_m \overrightarrow{a_0 a_m}$ *for a unique family* $(x_1, x_2 \cdots x_m)$ *of scalars called the co-ordinates of* x *with respect to the affine frame* $(a_0, (\overrightarrow{a_0 a_1}, \overrightarrow{a_0 a_2} \cdots \overrightarrow{a_0 a_m}))$.

Definition 2.7 5*Given two affine space* $< E, \vec{E}, +>$, *and* $< E', \vec{E'}, +' >$, *a function* $f: E \to E'$ *is an affine map if for every family* $(a_i, \lambda_i)_{i \in I}$, *of weighted points in* E *such that* $\sum_{i \in I} \lambda_i = 1$, *we have* $f(\sum_{i \in I} \lambda_i a_i) = \sum_{i \in I} \lambda_i f(a_i)$. *(i.e)* f *preserve barycenter.*

Definition 2.8 6 *Let* $A_1 \in R^n$ *and* $A_2 \in R^m$ *be two affine sets let* $f: A_1 \to A_2$ *a function.*f *is an affine transformation if and only if given subsets* $P_1, P_2, \ldots P_r$ *of* A_1 *any constants* $a_1, a_2, \ldots a_r \in R$ *for which* $\sum_{i \in I}^{r} a_i = 1$, *then if* $P = \sum_{i \in I}^{r} a_i P_i$ *we have* $f(p) = a_1 F(P_1) + \ldots + a_r F(P_r)$.

Definition 2.9 7 *Let* $f: A_1 \to A_2$ *be an affine transformation* $'f'$ *is invertible if we can find an affine transformation* $g: A_2 \to A_1$ *such that* $g \circ f = 1A_1$ *and* $f \circ g = 1A_2$.

III. SOME RESULTS

Proposition 3.1 8 *Let* $f: A_1 \to A_2$ *be an affine transformation of affine spaces and let* $X = P_0, P_1, \ldots P_s$ *be a set of points for which* $A_1 = AFf(X)$. *Then* f *is completely determined by its values at the points* $P_0, P_1, P_2, \ldots P_s$.

Proof: Since every point P in A_1 can be written as
$P = a_0 p_0 + a_1 p_1 + \ldots + a_s p_s$ with $\sum_{i=a}^{s} a_i = 1$ (or) $f(P) = \sum_{i=0}^{s} a_i f(P_i)$ that determines f(P).

Note: Like linear transformation, affine transformation on finite dimensional affine sets are completely determined by a finite amount of information.

Theorem 3.2 9 *Let* $f: A_1 \to A_2$ *be an affine transformation* f *is an invertible liner transformation iff* f *is both 1-1 and onto.*

Proof: The existence of g forces to be both 1-1 and onto.

The fact that f is both 1-1 and onto forces there to be a function g for which $g \circ f = 1A_1$ and $f \circ g = 1A_2$. This is standard fact about function.

The only thing we have really to prove is that the function which we exists is an affine transformation.

(i.e)Let $g = f^{-1}$.

Then we shows given $Q_1 \ldots Q_r$ in A_2 and $b_1, \ldots b_r$ in R with $\sum_{i=1}^{r} b_i = 1$ then if $Q = \sum_{i=1}^{r} b_i Q_i$.

we have $g(Q) = \sum_{i=1}^{r} b_i g(Q_i)$.

Since f is $1-1$ and onto we can find $P_1, \ldots P_r$ in A_1 such that $f(P_i) = Q_i$ (equivalently $g(Q_i) = P_i$).

Let $P = \sum_{i=1}^{r} b_i P_i$ then, since f is an affine transformation .

we have

$$f(P) = \sum_{i=1}^{r} b_i f(P_i)$$
$$= \sum_{i=1}^{r} b_i Q_i$$
$$= Q$$

Again

$$P = g(Q)$$

$$= \sum_{i=1}^{r} b_i P_i$$

$$= \sum_{i=1}^{r} b_i g(Q_i)$$

Theorem 3.3 10 *Let* V *and* W *be subspaces, respectively of* $\mathbb{R}^n$ *and* $\mathbb{R}^m$ *and let* $f: V \to W$ *be an affine transformation. If* $f(0) = 0$ *then* f *is linear transformation.*

Proof: We show for all v_1, v_2 in V we have $f(v_1 + v_2) = f(v_1) + f(v_2)$ and for all $v \in V$ and $\lambda \in R$.

We have $f(\lambda v) = \lambda f(v)$.

But notice that $1v_1 + 1v_2 - 1(v_1 + v_2) = 0$ and since $1 + 1 - 1 = 1$ and also since f is an affine transformation.

We have $1f(v_1) + f(v_2) - 1f(v_1 + v_2) = f(0) = 0$

(i,e) $f(v_1) + f(v_2) = f(v_1 + v_2)$

In a similar fashion, but this time noticing that $1(\lambda v) - \lambda(v) + \lambda(0) = 0$ and the fact that for only real number λ we have $1 -$

$\lambda + \lambda = 1$

$$\Rightarrow f(\lambda v) - \lambda f(v) + \lambda f(0) = f(0) \qquad \text{Since } f(0) = 0$$
$$f(\lambda v) - \lambda f(v) + \lambda 0 = 0$$
$$\text{(i,e) } f(\lambda v) = \lambda f(v).$$

Corollary 3.4 11 *Suppose that f is any affine transformation between vector spaces, as above and suppose that $f(0) = w$. Let $T_{-w}: W \to W$ be translation by the vectors $-w$ on the vector space W.*

$$\text{(i,e) } T_{-w}(w_1) = w + w_1$$

we already showed that T_{-w} is an affine transformation from W to W which is easily seen to be invertible.
Now consider $F = T_{-w} \circ f: V \to W$ since both of these are affine transformations so is their composition.
(i,e) F is an affine transformation,
Moreover $F(0) = (T_{-w} \circ f)(0) = T_{-w}(f(0)) = T_{-w}(w)$
$$= -w + w$$
$$= 0$$
(i,e) F is an affine transformation which takes '0' .

Theorem 3.5 12 *Let V and W be subspace of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Let $f: V \to W$ be an affine transformation then we can always find a translation of W such that $T \circ f$ is a linear transformation from V to W.*
Proof: If A is an affine subsets of $\mathbb{R}^n$ it is an easy fact that
$Aff(A) = \{f: A \to A / f$ is an invertible affine transformation$\}$
is a group under composition of function the identity of this group is the identity of affine transformation and the inverse of an invertible affine transformation is an generalized inverse of affine transformation.
(i,e) When A is a vector subspace of R^n.
we seen
(i)The invertible linear transformation are in this group.
(ii)The translation are in this group.
(iii)Everything in this group is the product of invertible linear transformation with a translation.

Theorem 3.613 *Let V be a subspace of $\mathbb{R}^n$ and let $Aff(v)$ be the group of invertible affine transformations $f: V \to V$ then*
(i)The translation of V call the set of them $T(v)$ are a subgroup of V isomorphic to $(V, +)$ as a group.
(ii)The invertible linear transformation from V to V, denoted by $Gl(v)$ is a subgroup of $Aff(v)$.
Proof: Let T_{v_1} and T_{v_2} be two translations the first by v_1 and second by v_2 then

$$(T_{v_1} \circ T_{v_2})(v) = T_{v_1}((T_{v_2})(v))$$
$$= T_{v_1}(v_2 + v)$$
$$= v_1 + (v_2 + v)$$
$$= (v_1 + v_2) + v$$
$$= T_{v_1+v_2}(v).$$
$$\text{(i,e)} \quad T_{v_1} \circ T_{v_2} = T_{v_1+v_2}.$$

The identity element is translation by the 0- vector and the inverse of the translation by v is the translation by $-v$.
$\Rightarrow$ The translations form a subgroup of invertible affine transformations of v.
Define function $\phi: V \to Aff(v)$ which takes $v \to T_v$.
"The invertible linear transformation on a vector space from a group under composition."
To show that the subgroup $Gl(v)$ is a normal subgroup of $Aff(v)$. If g is an element of $Aff(v)$, then $gGl(v)\, g^{-1} \subseteq Gl(v)$.
we know that if $g \in Aff(v)$ then $T_v \circ f = f$, where f is an invertible linear transformation and T_v is a translation.
$$\Rightarrow g = T_{-v} \circ f$$
And $$g^{-1} = f^{-1} \circ (T_{-v})^{-1}$$
But the inverse of T_{-v} is T_v.
$$\Rightarrow g^{-1} = f^{-1} \circ T_v.$$
But then if $h \in Gl(v)$ then
$$g \circ h \circ g^{-1} = (T_{-v} \circ f) \circ h \circ (g^{-1})$$
$$= (T_{-v} \circ f) \circ h \circ (f^{-1} \circ T_v)$$
$\Rightarrow$ which is a linear transformation on v and it takes the vector $'0'$ to itself.
$Gl(v)$ is a normal subgroup of $Aff(v)$.

Theorem 2.7 14*Let $Af(f): E \to E$ and $Af(g): E \to E$ be two affine maps with $a(n+1) \times (n+1)$ matrix represented by $Af(f)$ by A and $Af(g)$ by B then $Af(f)$ is invertible iff A is invertible.*
Poof: Let A be an invertible matrix.
Let $\vec{b}$ be a vector and Let $Af(f): E \to E$ defined on $\mathbb{R}^2$ via $\vec{x} \to A\vec{x} + \vec{b}$.

For any vector $\vec{y}$ we have $f(\vec{x}) = \vec{y}$.

$$A\vec{x} + \vec{b} = \vec{y}$$
$$A\vec{x} = \vec{y} - \vec{b}$$
$$A^{-1}(A\vec{x}) = A^{-1}(\vec{y} - \vec{b})$$
$$(A^{-1}A)(\vec{x}) = A^{-1}\vec{y} - A^{-1}\vec{b}$$
$$I_2\vec{x} = A^{-1}(\vec{y} - \vec{b})$$
$$\vec{x} = A^{-1}(\vec{y} - \vec{b})$$

(i,e)f^{-1} exits and can be given by $f^{-1}(\vec{x}) = A^{-1}(\vec{x} - \vec{b})$ and $(f^{-1} \circ f)(\vec{x}) = \vec{x}$
which is just an bijections of E defined on $\mathbb{R}^2$ and called as invertible linear transformation with respect to $\vec{x}$.
Similarly on the same way B be an invertible matrix.
Let $\vec{v}$ be a vector and let $Af(g): E \to E$ defined on $\mathbb{R}^2$ via. $\vec{t} \to B\vec{t} + \vec{v}$
For any vector $\vec{s}$ we have $g(\vec{t}) = \vec{s}$

$$B\vec{t} + \vec{v} = \vec{s}$$
$$B\vec{t} = \vec{s} - \vec{v}$$
$$B^{-1}(B\vec{t}) = B^{-1}(\vec{s} - \vec{v})$$
$$B^{-1}B(\vec{t}) = B^{-1}\vec{s} - B^{-1}\vec{v}$$
$$I_1\vec{t} = B^{-1}\vec{s} - B^{-1}\vec{v}$$
$$\vec{t} = B^{-1}(\vec{s} - \vec{v})$$

(i,e) g^{-1} exits and can be given by $g^{-1}(\vec{t}) = B^{-1}(\vec{s} - \vec{v})$ and $(g^{-1}\vec{g})(\vec{t}) = \vec{t}$.
which is again a bijection and called a invertible linear transformation .
Similarly $f \circ g$ the composition of affine transformation represented by the matrix $f: E \to E$ and $g: E \to E$ with $\begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} B & c \\ 0 & 1 \end{bmatrix}$ with respect to the affine frame.
In which f is invertible $\Leftrightarrow$ A is invertible, g is invertible $\Leftrightarrow$ B is invertible.
$f \circ g$ is defined as $\begin{bmatrix} AB & Ac + b \\ 0 & 1 \end{bmatrix}$ which also invertible represented as
$f^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}b \\ 0 & 1 \end{bmatrix}$, $g^{-1} = \begin{bmatrix} B^{-1} & -B^{-1}c \\ 0 & 1 \end{bmatrix}$
and the solution exists in a unique way.

REFERENCES

[1] G.Radhakrishna Rao, *" A note on the Generalized inverse of a Matrix with application to Problems in Mathematical statistics"[Journal of Royal Statistical Society]* Vol. 24, No. 1(1962) pp.152-158.

[2] P.K.Suetin,*" Linear Algebra and Geometry,(Algebra and Logic and Application)," * first edition 1989, [Crc . pr. Inc].

[3] Jean H.Gallier, "*Curve and surface in Geometric modelling,"* first edition 1999.

[4] Gilbert strang *"Linear Algebra and its application,"* Third edition 1988.

[5] Jean.H.Gallie, *"Geometric method and Application for Computer science and Applications,"* Second edition 2011.

[6] Stephen L.Campbell Carl D.Meyer , *"Generalised inverse of Linear Transformation,"* 2009.

[7] Kenneth Hoffman and Kunze, *"Linear Algebra and its application,"* Second edition 1971, PRENTICE-HALL, INC., Englewood Cliffs, New Jersey.