THE SYLOW THEOREMS

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***Abstract*: This paper introduce the concept of group actions, along with the associated notions of orbits and the Sylow theorem, we build up to the three Sylow theorems, and subsequently give some of their examples.**

***Index Terms*: p-subgroup, normal subgroup, proper subgroup, Finite Group.**

I. INTRODUCTION

In a group theory Lagranges theorm is one of the important theorem in a finite groups, which states that the order of any subgroup of a group divide the order of the group. But the converse of this theorem is not true, that for any number dividing the order of a group, there exists a subgroup of that order. The Sylow theorems do gives us partial converse to Lagrange's theorem, by asserting the existence of certain subgroups (called Sylow p-subgroups) of any group with a given order, and gives some information about their properties. In this paper, we wish to build up to the knowlege of the Sylow theorems. A group G is said to be a p-group(p is prime number), if the order of every element of G is some power of p.

SYLOW THEOEM : First Sylow theorem States : If p^m divides the order Of a finite Group G(p being a prime number), then G has a subgroup of order p^m.We know that the group is simple if it has no proper normal subgroups. It is applied to determine whether a given finite group is simple or not.

Theorem: If p is a prime number and pm divides o(G) , then G has subgroup of order pm.

Note: A particular case of the above theorem is the following: If p is a prime number such that p^m/o(G) and p^{m+1}†o(G) , then G has a subgroup of order pm.. Sylow p- subgroup (Definition) : If G is the finite group and p is a prime number, then a subgroup of G of order pm , then a subgroup of G of orderpm, where pm/o(G) but pm+1†o(G) is called sylow p- subgroup or p-sylow subgroup of G or in short p-SSG of G.

Note: The above definition implies that all sylowp- subgroups of G are of the same order i.e. p^m. Since any p-subgroup of G is of order, a power of p, it follows that, No subgroup of G can contain properly a sylow p-subgroup of G.

Note: By the above statement, it follows that A finite group G has a p- sylow subgroup for each prime p which divides o(G). **Remark**: The condition pm/o(G)andp^{m+1}†o(G) imply that m is the highest power of p such that p^n/o(G) . Equivalently the above conditions enable us to write o(G) as follows :

$$O(G) = p^m.n, \text{ where } p†n$$

The first sylow theorem can be restates as:

If o(G) = p^m.n , then G has a sylow p-subgroup of order pm. This form is helpful in finding sylow p-subgroup of a finite group.
Example: i) Of o(G) = 48 = 16*3=24*3, Where 2†3; then G has a 2-SSG of order 24= 16. Also G has a 3- SSG of order 3.

ii) If o(G) = 56 = 23* 7, then G has a 2-SSG of order 23= 8 and a 7-SSG of order 7.
Note : From the above we note that , The first sylow theorem tells us as to which type of p-sylow subgroup a given finite group possesses.

Lemma (Double Coset Decomposition): If A and B are two subgroup of a group G, then G = $\bigcup_{x\in G} AxB$

Lemma: If A and B are finite subgroups of a group G, then

$$G = \bigcup_{x\in G} AxB \quad \text{Then } o(AxB) = \frac{o(A)o(B)}{o(A\cap xBx^{-1})}, \quad x\in G \quad ,$$

Second Sylow Theorem : Any two sylow p-subgroup of a finite group G is unique if and only if it is normal.
If H is the only p-SSG of G, then H is normal in G.

Lemma: Let P be a sylow p- subgroup of a finite group G. then the number n_p of Sylow p-subgroup of G is given by n_p = 1+kp And n_p divides o(G).

Note : The first Sylow theorem tells us as to what type of Sylow p-subgroups, a finite group G can have. The Third Sylow theorem will tell us as to how many sylow p-subgroups, G can have.

Third Sylow Theorem : show that the number n_p of Sylow p-subgroup of a finite group G is given by n_p= 1+ kp,

Where k = 0, 1, 2,… and n_p divides o(G).

Rule: The number (n_p) of p-SSG of G is given by n_p=1+ kp ; k = 0, 1, 2,… and n_p divides o(G).

Remark: The formula for k = 0 is always true, for n_p = 1 and1 divides o(G). The problem becomes more interesting if we find some positive integral value of k such that n_p= 1+kp and np divides o(G).

First sylow theorem: If G is finite group such that pn/o(G) and pm+1†o(G) (p being prime) then G has a subgroup H of order p_n. H is called a Sylow p- subgroup of G.

Second Sylow theorem: Any two Sylow p-Subgroups of a finite group G are conjugate in G i.e. If P and Q are two sylow p-subgroup of G then Q = xpx-1for some

Third Sylow theoem: The numbernp of sylow p-subgroup of a finite group G is given by np=1+ kp ; k = 0, 1, 2,… and np divides o(G).

Example: Prove that the group of order 28 has a group of order 7. And hence prove that a group of order 28 is not simple.

Solution: We have o(G) = 28 = 7*22By First sylow theorem , 7-SSG is given by n_7= 1 + 7k, k = 0, 1, 2,… and n7 divides o(G) = 28.

For k=0, n_7= 1 and n7 divides order of G =28 .

For k= 1,n_7 =8 but 8 does not divide o(G)= 28.

For k= 2,n_7 =15 but 15 does not divide o(G)= 28. And so on.

Therefore, n_7= 1 i.e. there exist exactly one7-SSG say H, where, o(H) = 7. Hence H is a normal Subgroup of order 7. Recall that group G is simple if it has no proper normal subgroup.

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