© 2019 IJRTI | Volume 4, Issue 11 | ISSN: 2456-3315

Solution of Viscous Flows in Driven Cavity by Efficient Numerical Techniques

Dr. Rajesh Kumar

Assistant. Professor (S.G.)
Department of Mathematics,
D.A.V. (P.G.) College, Dehra Dun, India.

***Abstract*: The motion of a liquid under action of viscous forces is governed by Navier - Stokes equations. Driven cavity problem has worked as an ideal prototype. There are still continuously increasing interest of researchers in the driven cavity flow problem with different surface twists. The number of numerical methods have been tried on this problem and results have been published by authors. We have also applied our technique for the solution of this classical problem and obtained results which are comparable with the previous results are in good agreement.**

Index Terms: Numerical Methods, Navier-Stokes Equations

I. INTRODUCTION:

From a computational viewpoint, the cavity flow is an ideal prototype non-linear problem which is readily posed for numerical solution. Because of its geometrical simplicity and comparatively minor singularities, it has served as a model problem for last twenty years for testing new numerical schemes and as a benchmark solution for making comparisons among various schemes. There are still continuously increasing interest of researchers (22),(23) in the driven cavity flow problem with different surface twists for applying numerical methods. Fortunately, a number of numerical methods have been tried on this problem and results have been published. We have also applied our technique for the solution of this classical problem and obtained results which are comparable with the previous results.

The cavity, as shown in fig.(1) is rectangular in cross- section and filled with a Newtonian, viscous and incompressible fluid. The fluid is forced to make by the motion of the upper surface which travels with constant linear velocity in its own plane. The cavity is assumed to be long in the longitudinal (Z) direction so that the fluid motion is essentially two dimensional.

II. THE PROBLEM:

The equations for two dimensional steady flow may be written asbelow. The Navier-Stokes equations can be formulated in terms of vorticity (ω) and stream function (ψ) as :

$\nabla^2\omega - \mathrm{Re}(\psi_y\omega_x - \psi_x\omega_y) = 0$ (1)

$\nabla^2\psi = -\omega$ (2)

Satisfying

$$\psi_y = \frac{\partial\psi}{\partial y}, \psi_x = \frac{\partial\psi}{\partial x}$$

and

$$\omega_x = \frac{\partial\omega}{\partial x}, \omega_y = \frac{\partial\omega}{\partial y}$$

Where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

These equations (1) and (2) can be formulated as follows:

$$\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2} - \mathrm{Re}\left\{\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}\right\} = 0 ---------- (3)$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -\omega ------------------------(4)$$

The numerical solution procedure generally consists of discretizing the domain over which equations (3) - (4) are defined. This gives two systems of algebraic equations. One can obtain the solution in terms of stream function and vorticity.These equations together with boundary conditions constitutes a non- linear elliptic boundary value problem. The degree of non- linearity increases with Reynolds number.

III. METHEMATICAL MODEL FOR DRIVEN CAVITY PROBLEM:

The cavity, as shown in fig.(1) is rectangular in cross- section and filled with a Newtonian, viscous and incompressible fluid. The fluid is forced to make by the motion of the upper surface which travels with constant linear velocity in its own plane. The cavity is assumed to be long in the longitudinal (Z) direction so that the fluid motion is essentially two dimensional.

Burggraf (1) was first to employ central difference scheme for the solution of equations (3) - (4) . The resulting systems of equations were solved by using iterative procedures. But Iterative procedures fail to converge even at moderate Reynolds number. This difficulty originated the idea of upwind schemes. However, it is well known that upwind differencing introduces false diffusion effects and this will lead to additional error into the numerical solution as shown by Strikwerda (2).

The formation of the governing equation for steady motion has been described in detail by Mills (6) and Burggraf (1). The steady flow in the square cavity (fig. (2)) is governed by the equations (3) & (4) together with boundary conditions.

IV. DERIVATION OF BOUNDARY CONDITIONS:-

We compute these with the help of derivative boundary conditionsonψ. Assuming equation (3) is valid near the wall and using n to represent the coordinate normal to the wall.

$$\omega(1) = -\left(\frac{\partial^2\psi}{\partial n^2}\right)_1 \text{-----------------------} (5)$$

Where the argument and subscript '1' denote the mesh point on the wall. Using a Taylor's series expansion for ψ with step length Δn about the mesh point on the wall.

$$\psi(2) = \psi(1) + \Delta n\left(\frac{\partial\psi}{\partial n}\right)_1 + \frac{\Delta n^2}{2}\left(\frac{\partial^2\psi}{\partial n^2}\right)_1 + \frac{\Delta n^3}{6}\left(\frac{\partial^3\psi}{\partial n^3}\right)_1 + O(\Delta n^4) \text{--------} (6)$$

$$\psi(3) = \psi(1) + 2\Delta n\left(\frac{\partial\psi}{\partial n}\right)_1 + \frac{4\Delta n^2}{2}\left(\frac{\partial^2\psi}{\partial n^2}\right)_1 + \frac{8\Delta n^3}{6}\left(\frac{\partial^3\psi}{\partial n^3}\right)_1 + O(\Delta n^4) \text{-----} (7)$$

From (6) and (7) and making use of (5) , we get

$$\omega(1) = \left[\psi(3) - 8\psi(2) + 7\psi(1) + 6\Delta n\left(\frac{\partial\psi}{\partial n}\right)_1\right] / (2\Delta n^2)$$

Since $\psi(1)$ is zero on all four boundaries, the above equation reduces to

$$\omega(1) = \left[\psi(3) - 8\psi(2) + 6\Delta n\left(\frac{\partial\psi}{\partial n}\right)_1\right] / \left[2\Delta n^2\right] \text{---------------------} (8)$$

This is second order approximation for vorticity and was used by Bozeman and Dalton (4). Roache (3) calls it Jensen's fourmula. It is also referred as Briley's formula. Gupta and Manohar (5) have investigated the effects of boundary approximations on the solution and shown that Briley's formula gives more accurate results than any of the other formula.

The boundary conditions for ψ may be derived from the conditions that each wall is impermeable and that the viscous boundary conditions implies that the fluid adjacent to the wall moves with the same velocity as the wall. The first condition implies that $\psi=0$ on each boundary and second that the derivative of ψ in direction normal to the wall is equal to the tangential velocity of the wall. The boundary conditions thus obtained for driven cavity problem are as under:

$\psi=0$ along all boundaries ---------------(9)
$\psi_x=0$ along vertical walls OC & AB ---------(10)
$\psi_y=0$ along the bottom wall OA ----------- (11)
$\psi_y=-1$ along the sliding wall BC ----------- (12)

V. EARLIER SOLUTIONS OF THE PROBLEM:-

Burggraf (1) was first to employ central difference schemes for this problem using iterative procedures. Since the diagonal dominance property is not nacessarily satisfied, the standard iterative procedures such as Gauss- Seidel or SOR fail to converge rapidly even at moderate high Reynolds number. Later Spalding (7) discovered that stable solution could be obtained if upwind differencing methods were used for the convective terms in the vorticity equation. Further, Finite differnce calculations using similar ideas have been presented by Nallaswamy and Krishna Prasad (8). However, recently Strikwerda (2) and others have brought out clearly the drawbacks of upwind schemes.

The second technique to overcome the convergence problem is that at each outer iteration, the algebraic equations corresponding (3) -(4) are solved by direct fast solvers. Gupta (9) has applied this approach and used direct solver MA28 from the Harwell package (10). But these solvers do not take the benefit of sparsity into consideration and thus are not economical in computer- time and storage.

Roache (3) gave the idea of what he calls Laplacian Driver Method. It is worth emphasizing that the superiority of Laplacian Driver Method over other methods is its simplicity to apply on computer, because one has to solve two Poisson's equations iteratively. We have applied an elliptic equation solver to solve equation (3) and Poisson's equation solver for equation (4). The system of equation (3) and (4) is solved iteratively until a desired convergence- criterion is satisfied.
Roache (3) has reported the solution of the driven cavity flow problem for Reynolds number Re=20 with a 11x11 mesh. By using our technique, we have obtained the solutions upto Reynolds number Re=500 with a 21x21 mesh. The results obtained by us compare well with those of previous authors.

VI. CONVERSION TO DIFFERENC EQUATIONS:

The rectangular region over which the equations (1) & (2) are defined is divided into a uniform mesh by choosing mesh widths h along x-direction and k along y- direction. Applying central difference formulae to (1) & (2), relative to a uniform

$$A_1\omega_{i,j-1} + A_2\omega_{i-1,j} + A_3\omega_{i,j} + A_4\omega_{i+1,j} + A_5\omega_{i,j+1} = A_6 ------(13)$$

Where

mesh system , we obtain:

VII. COMPUTATIONAL METHOD:-

The resulting structure of system of equation (13) -(14) can be rewritten as :
AU=V ---------------------------------- (15)

$$A_1 = \frac{1}{k^2} + \frac{\text{Re}}{4hk}\left(\omega_{i+1,j} - \omega_{i-1,j}\right)$$

$$A_2 = \frac{1}{h^2} + \frac{\text{Re}}{4hk}\left(\omega_{i,j+1} - \omega_{i,j-1}\right)$$

$$A_3 = -2\left(\frac{1}{h^2} + \frac{1}{k^2}\right)$$

$$A_4 = \frac{1}{h^2} + \frac{\text{Re}}{4hk}\left(\omega_{i,j+1} - \omega_{i,j-1}\right)$$

$$A_5 = \frac{1}{k^2} + \frac{\text{Re}}{4hk}\left(\omega_{i+1,j} - \omega_{i-1,j}\right)$$

AND

$$A_6 = 0$$

$$B_1\psi_{i,j-1} + B_2\psi_{i-1,j} + B_3\psi_{i,j} + B_4\psi_{i+1,j} + B_5\psi_{i,j+1} = B_6 ---------(14)$$

Where

$$B_1 = \frac{1}{k^2}$$

$$B_2 = \frac{1}{h^2}$$

$$B_3 = -2\left(\frac{1}{h^2} + \frac{1}{k^2}\right)$$

$$B_4 = \frac{1}{h^2} \qquad B_5 = \frac{1}{k^2}$$

$$B_6 = -\omega_{i,j}$$

Coefficient matrix A is a block tri-diagonal system having n blocks where each block is of order (mxm). U is a column matrix of unknown and V is also a column matrix containing right hand sides of equations. Since diagonal dominance property in such a system does not always hold, so iterative methods sometimes fail to converge. Recently some efficient direct methods have been employed to solve such a system. Linger (11) has applied a semi - direct method to solve Poisson's equation.

Elsner and Mehramann (12) have dealt in detail on the convergence condition of block iterative methods. The block tri-digonal system of linear equations is a sparse coefficient matrix system and it is possible to take the advantage of the sparseness in order to reduce both computation time and storage requirements. Duff (13), Erisman and Reid (14) have dealt in detail with the direct methods for sparse matrices . Jennings (15) has also given methods like elimination using submatrices to deal with a sparse structure of such type.

The algorithm also employs a direct elimination technique to solve block tri-diagonal system of linear equations. Important steps in brief are as follows. Detailed analysis of algorithm and storage etc. are given in Sharma and Agarwal (16) .

Following are the steps involved in the computational procedure.

(1) assign initial approximation $\psi^{(m)}$ with m=0, Here approximation is taken as $\psi^{(0)}=0$.

(2) Compute boundary values for ωby boundary conditions as in (8).

(3) Solve for Vorticity $\varpi^{(m+1)}$ from

$$A_1\omega_{i,j-1}^{m+1} + A_2\omega_{i-1,j}^{m+1} + A_3\omega_{i,j}^{m+1} + A_4\omega_{i+1,j}^{m+1} + A_5\omega_{i,j+1}^{m+1} = A_6$$

(4) Damp values of Vorticity in interior of domain as follows

$\omega^{(m+1)} = \delta\varpi^{(m+1)}+(1-\delta)\,\omega^{(m)}$, $0<\partial<1$

$$B_1\psi_{i,j-1}^{m+1} + B_2\psi_{i-1,j}^{m+1} + B_3\psi_{i,j}^{m+1} + B_4\psi_{i+1,j}^{m+1} + B_5\psi_{i,j+1}^{m+1} = B_6$$

(5) Solve stream function equation to obtain $\psi^{(m+1)}$ as

(6) Damp values of stream function as

$\psi^{(m+1)} = \delta\psi^{(m+1)} + (1-\delta)\psi^{(m)}$

(7) Repeat steps (2) - (6) until following convergence criterion is satisfied.

$$\max \left| \omega^{(m+1)}{}_{i,j} - \omega^{(m)}{}_{i,j} \right| < \epsilon$$

(8) After above convergence criterion is satisfied , boundary values for ω are again computed as in step (2) and solution is attained over whole domain.

(9) The results are taken as computed for number of iteration for the convergence and precision.

At each iteration , system of linear equations is solved by a specially designed algorithm [16] which is economical in computer storage and computer time both. It also provides accurate solutions.

VIII. RESULTS AND DISCUSSIONS:

The point at which the value of ψ attains it's absolute maximum is called the center of primary vortex (vc). Values of ψ and ω at the vortex center are by ψ_{max} and ω_{vc} respectively and a uniform mesh-size of h=1/20 has been taken for different Reynolds numbers.

The results obtained here are compared well with those of previous authors [(5),(8),(9),(18),(19),(20),(21)] as shown in Table-I and Table II. Streamlines and eqivorticities curves for different Reynolds number have been drawn. From Figures it is clear that there is no secondary vortex at Re=1 and 10, but there exists two secondary vorticities at the downstream corners for Re=100 to Re=500. Also the size of secondary vortices increases with the increase in Reynolds number as observed experimentally by Pan and Acrivos (17), Uddin and Saha (24) . The equivorticity curves become more asymmetrical and recirculating eddies become more dominant with the increase in Reynolds number. The equivorticity curve at Re=500 has a secondary eddy on the bottom wall at the level -1.0. The same nature of vorticity curve was observed by Ghia, Ghia and Shin (18) at Re =400 using a much finer mesh.

IX. CONCLUSIONS: We may conclude from the results discussed that-

1. Present scheme and algorithm works well up to moderately large Rayleigh numbers.
2. Results obtained are in good agreement with previous authors.
3. Figures are showing the likely behavior as shown by previous authors (15).

Table –I

Values of Primary Vortex (ψ_{max})for different Reynolds Nos.

Re	(x,y)	ω_{vc}	ψ_{max} obtained	Earlier Results	Ref.	Mesh Size	Remarks
1	(0.35,0.75)	3.3553	0.0978	ψ_{max} =0.0982 =0.0995 =0.0995 =0.1082	(5) (5) (9) (5)	h=1/20 h=1/20 h=1/20 h=1/20	
10	(0.35,0.75)	3.3496	0.1130	ψ_{max} =0.1043 =0.1095	(19) (5)	h=1/30 h=1/20	
100	(0.35,0.75)	3.3496	0.113	ψ_{max} =0.1043 =0.1095	(19) (5)	h=1/30 h=1/20	
400	(0.40,0.60)	2.2363	0.1027	ψ_{max} =0.1129 =0.1139 =0.0970	(20) (18) (21)	h=1/40 h=1/30 h=1/20	
500	(0.45,0.60)	2.0132	0.1009	ψ_{max} =0.1024	(9)	h=1/20	

Table –II

Values of Vorticity at the Vortex center (ω_{vc}) for for different Reynolds Nos.

Re	(x,y)	Obtained ω_{vc}	Earlier Results	Ref.	Mesh Size	Remarks
1	(0.35,0.75)	3.3553	ω_{vc} =2.95 =3.02 =3.02 =3.33	(5) (5) (9) (5)	h=1/20 h=1/20 h=1/20 h=1/20	
10	(0.35,0.75)	3.3496	ω_{vc} =3.570 =3.155	(5) (8)	h=1/20 h=1/50	
100	(0.35,0.75)	3.3496	ω_{vc} =3.570 =3.155	(5) (8)	h=1/20 h=1/50	
400	(0.40,0.60)	2.2363	ω_{vc} =2.2810 =2.2947 =2.3600	(20) (18) (21)	h=1/40 h=1/30 h=1/20	
500	(0.45,0.60)	2.0132	ω_{vc} =2.0504 =1.9048	(9) (9)	h=1/20 h=1/20	

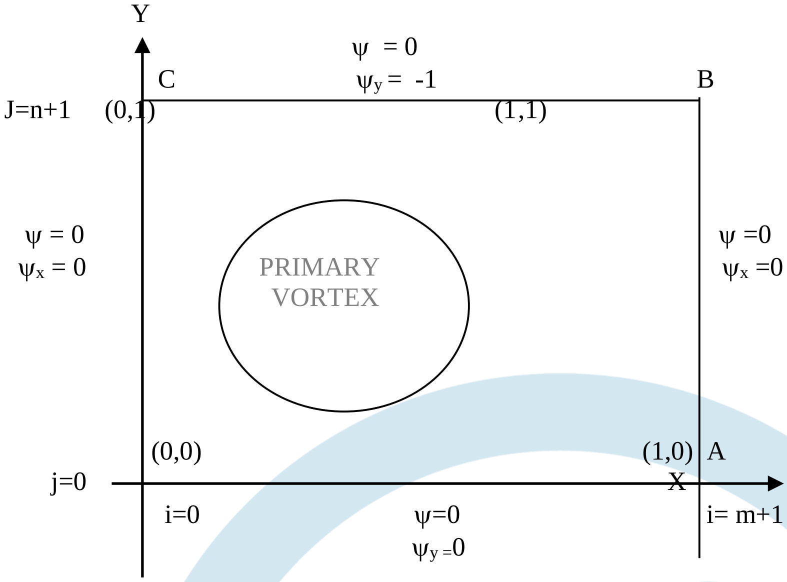


FIG. (1) – SQUARE DRIVEN CAVITY FLOW PROBLEM

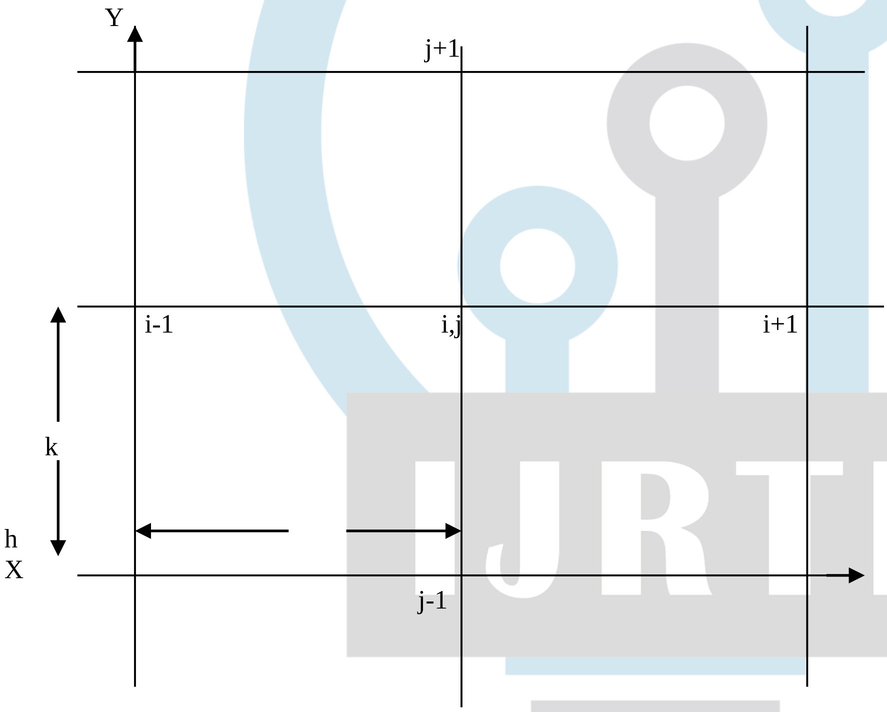
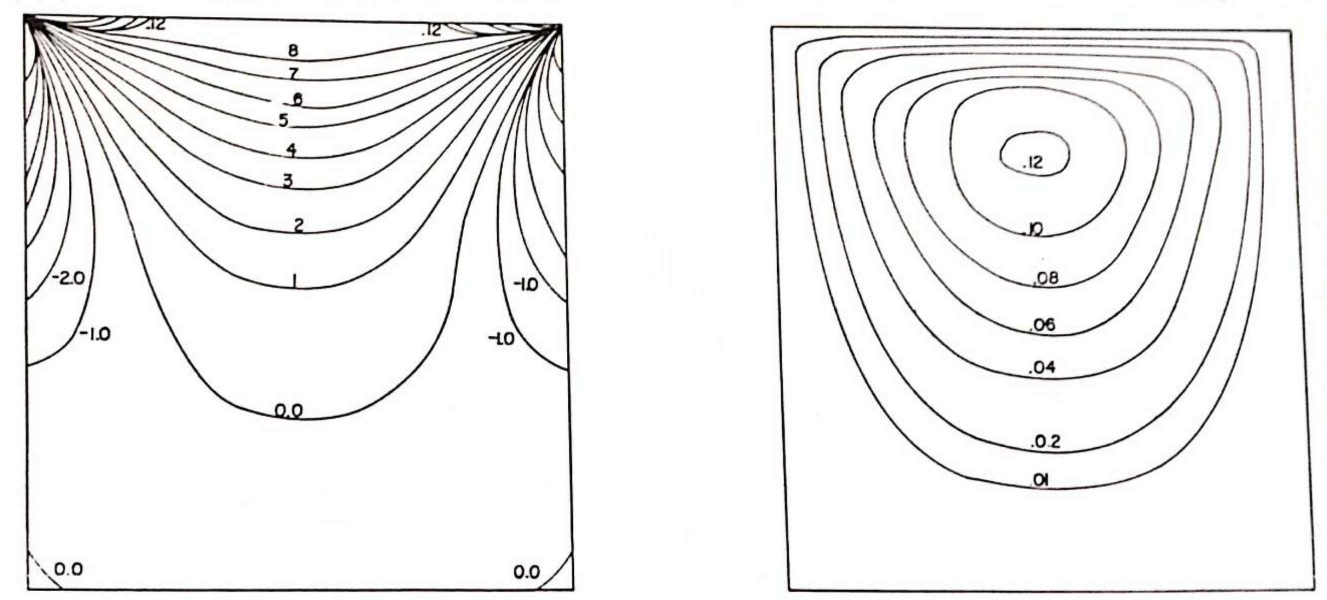
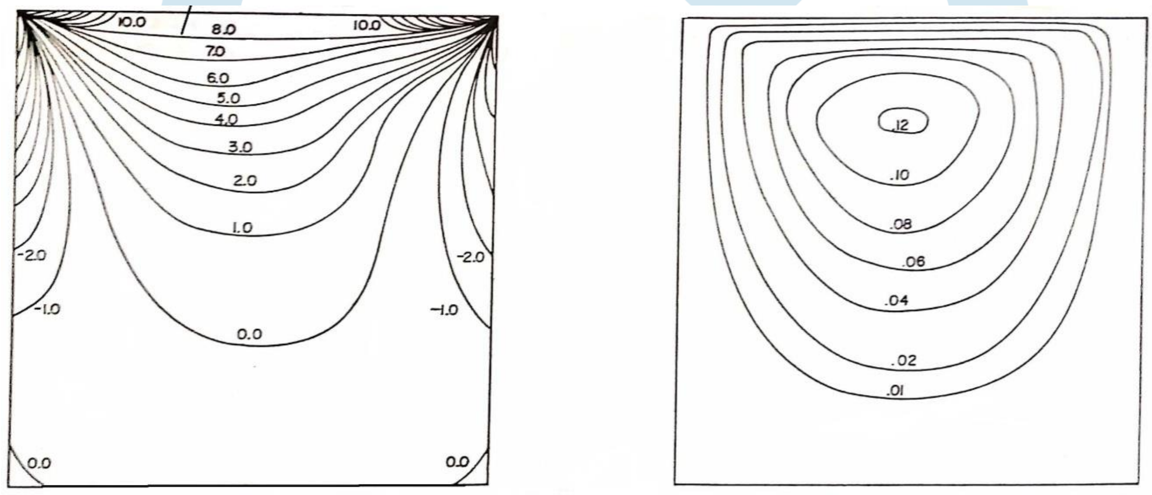


FIG.(2)- COMPUTATIONAL MOLUCULE FOR AN INTERIOR NODE

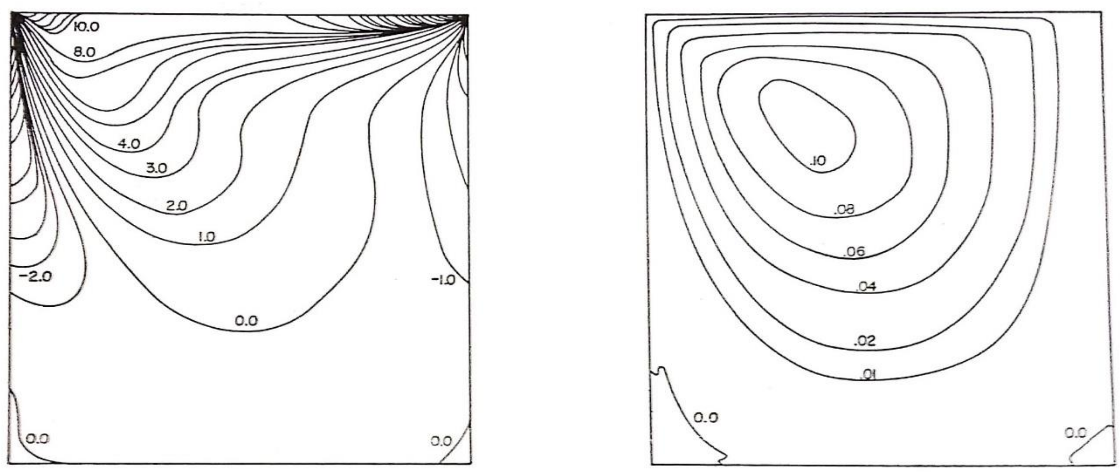
Vorticity curves for different Reynolds Nos. **Streamlines for Different Reyonolds Nos.**



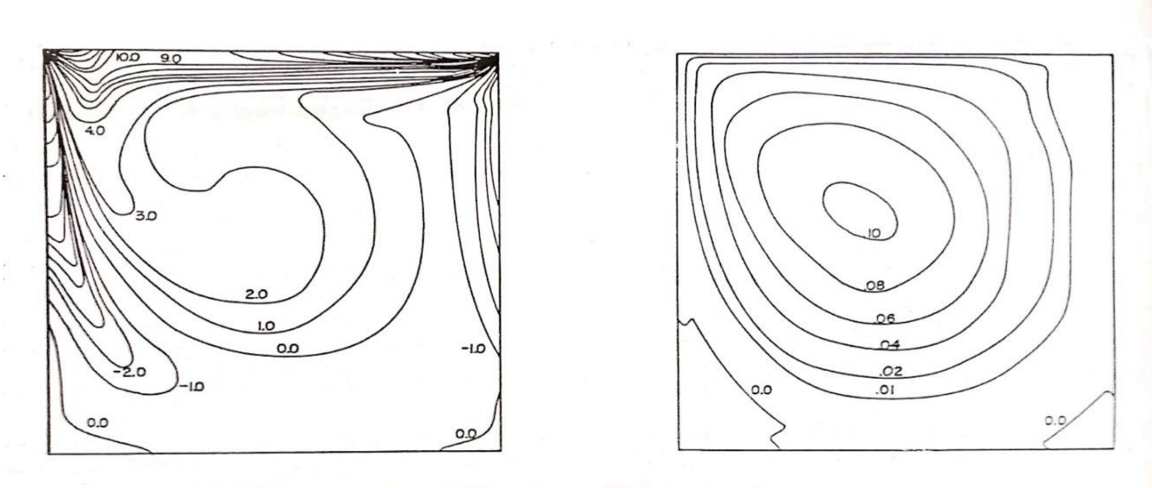
Re=1



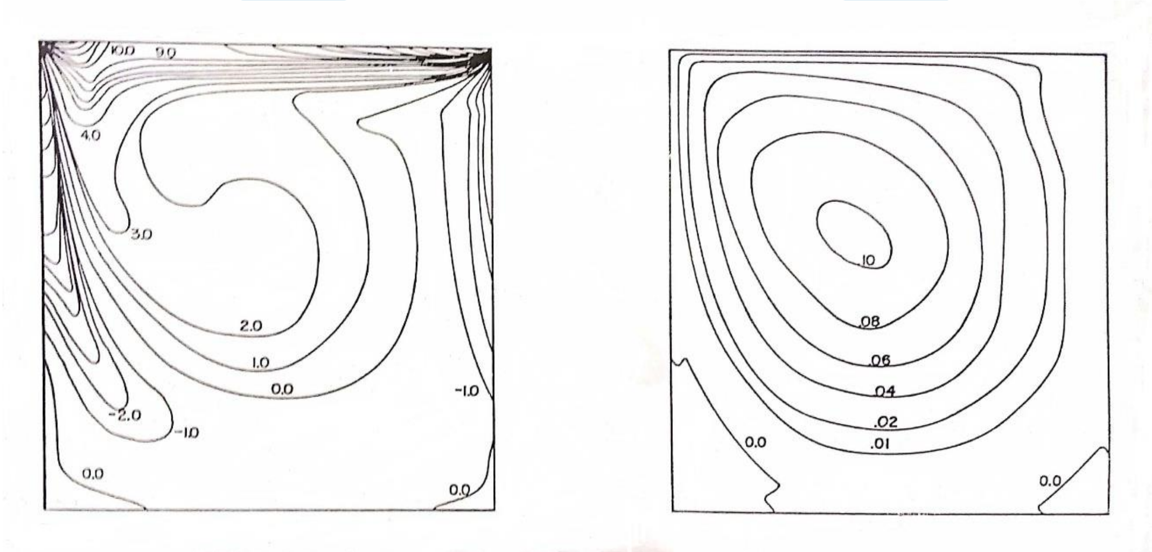
Re=10



Re=100



Re=400



Re=500

REFERENCES:

[1]. Burgraf O.R.,Analytical and Numerical studies of the structure of steady separated flows, J.Fluid Mech., 24, 113-151,1966
[2]. Strikwerda, J.C.,Upwind differencing, False scaling and non-physical solutions to the driven cavity problem, J. Comp.Phys.,47,303-307,1982.
[3]. Roache, P.J., The LAD, NOS and split NOS methods for the steady state Navier-Stokes equations, Comp. Fluids, 3, 179-195 ,1975.
[4]. Bozeman, J.D. and Dalton, C., Numerical study of viscous flow in a cavity, J. Comp. Phys., 12, 348-363 ,1973.
[5]. Gupata, M.M. and Manohar, R.P., Boundary approximations and accuracy in viscous flow computations, J.Comp. Phys., 31, 265-287 (1979).
[6]. Mills, R.D., On the closed motion of a fluid in a square cavity, J. Roy. Aero. Soc., 69,116-120,1965.
[7]. Spalding, D.B., A novel finite difference formulation for differential expressions involving both first and second derivatives, Int. J. Num. Meth. Eng., 4, 551-559 ,1972.
[8]. Nullaswamy, M and Krishna Prasad, On cavity flow at high Reynolds numbers, j. Fluid Mech., 79, 391-414 ,1977 .
[9]. Gupta, M.M., A comparision of Numerical solution of convective and divergence forms of the Navier-Stokes equations for the driven cavity flow problem, J. comp. Phys.,43,260-267, 1981.
[10] I.S. Duff, MA28- A set of FORTRAN dubroutines for sparse unsymmetric linear equations, report AERE-R8730,A.E.R.E. Hrwell, U.K., 1977.
[11] Linger,W., A quassi-direct fast Poisson solver for general regions, J. Commput. Appl. Math., 25-47,1999.
[12]. Elsner, N. and Mehramann, V., Convergence of block iterative methods for linear systems arising in the numerical solution of Euler equations, Numer. Math., 59, 541-559, 1991.
[13] Duff, I., Erisman, A. and Reid, J., Direct methods for sparse matrices, Oxford university Press, 1987.
[14]. Erisman, A. and Reid, J., Direct methods for sparse matrices, Oxford university Press, 1981
[15]. Jennings, A., Matrix computations for Engineers and Scintists, John Wiley and sons, 1977.

[16] Sharma, P.K. and Agarwal, M.K. : An economic elliptic equation solver for rectangular regions, accepted for publication in Vol. 10, J. of Engg. and Appl. Sci., Saudi Arabia.
[17]. Pan, F. and Acrivos, A., Steady flows in rectangular cavities, J.Fluid Mech., 28, 643-655 ,1967.
[18]. Ghia, U., Ghia,K.N and Shin, C.T., High re-solution for incompressible flow using the Navier-Stokes equations and a multi grid method, J. Comp. Phys., 48, 387-411 ,1982.
[19]. Rubin, S.G. and Graves, R.A., Viscous flow solutions with a cubic spline approximation, Comp. Fluids, 3, 1-36, 1975.
[20]. Schreiber, R. and Keller, H.B., Driven cavity flows by efficient numercal techniques, J. Comp. Phys., 49,310-333, 1983.
[21]. Barret, K.E. and Demunshi, G., Finite element solutions of convective diffusion problems, Int.J. Num. Meth. Eng. 14, 1511-1524, 1979.
[22]Shankar, P N, Deshpande, M.D., Fluid Mechanics in the Driven cavity, Ann. Rev. Fluid Mech. 32,93-136, 2000.
[23] Ercan Erturk, Bahtiyar Dursun, Numerical Solutions of 2-D Steady incompressible flow in a Driven skewed cavity, Journal of Applied Mathematics and Mechanics, Vol. 87,377-392, 2007.
[24] Salah Uddin, K.M., Litan Kumar Saha, Numerical Solution of 2-D Driven cavity flow with wavy bottom surface, American Journal of Applied Mathematics, 3(1-1), 30-42, 2015.