RP-136: Formulation of solutions of a standard cubic congruence of even composite modulus-an even-multiple of power of three

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***Abstract*: In this paper, a standard cubic congruence of even composite modulus –an even multiple of power of three is considered for discussion. After rigorous study the Formulations of the solutions of the said congruence are established. It is found that the Cubic congruence under consideration have exactly three solutions for an odd and Even positive integer. It has nine incongruent solutions for $a = 3$; and has twenty seven incongruent solutions for other multiples of three. The Author's formulations make It easy to find the required solutions. Formulation is proved time-saving.**

***Keywords*: Composite modulus, cubic residues, Standard cubic Congruence, Formulation.**

INTRODUCTION

Here the author's aim is to establish formulations of solutions of standard cubic congruence of composite modulus- an even multiple of power of three. It is the congruence of the type: $x^3 \equiv a^3 \pmod{2.3^n}$. Such congruence are always solvable.

A very little material is found in the literature of mathematics about the solutions of a standard cubic congruence of composite modulus. A cubic congruence of prime modulus is only defined by Koshy [2]. Zuckerman also had defined a cubic residue [3]. But no formulation is found for the cubic congruence of composite modulus except the author's formulation on some standard cubic congruence of even composite modulus [4],
[5], [6].

PROBLEM STATEMENT

Here, the problem under consideration is to find a formulation of the solutions of the Standard cubic congruence of even composite modulus of the type:

$X^3 \equiv a^3 \pmod{2.3^n}$ in two different cases:
Case – I: When $a \neq 3l$;
Case – II: When $a = 3l; l = 1, 2, 3 \dots \dots \dots$

ANALYSIS AND RESULT

Case-I: Let $a \neq 3l$.

Consider the cubic congruence: $x^3 \equiv a^3 \pmod{2.3^n}$.

For the solutions, let us consider $x \equiv 2.3^{n-1}k + a \pmod{2.3^n}$, $k = 0, 1, 2 \dots \dots \dots$

Then $x^3 \equiv (2.3^{n-1}k + a)^3 \pmod{2.3^n}$

$\equiv (2.3^{n-1}k)^3 + 3.(2.3^{n-1}k)^2.a + 3.2.3^{n-1}k.a^2 + a^3 \pmod{2.3^n}$

$\equiv 2.3^n k\{a^2 + 2.3^{n-1}ka + 2^2.3^{2n-3}k\} + a^3 \pmod{2.3^n}$

$\equiv a^3 \pmod{2.3^n}$, for a for any positive integer.

Thus, $x \equiv 2.3^{n-1}k + a \pmod{2.3^n}$ satisfies the said congruence.

Hence, it is a solution of the congruence.

But for $k = 3$, the solution becomes $x \equiv 2.3^{n-1}.3 + a \pmod{2.3^n}$

$\equiv 2.3^n + a \pmod{2.3^n}$

$\equiv a \pmod{2.3^n}$

Which is the same as for the solution for $k = 0$.

For $k = 4, 5, \dots\dots$ the solutions are also the same as for $k = 1, 2 \dots\dots$

Thus, the congruence has exactly three incongruent solutions

$x \equiv 2.3^{n-1}k + a \pmod{2.3^n}$, for any integer a, $k = 0, 1, 2$.

But for $a = 3$, consider $x \equiv 2.3^{n-2}k + 3 \pmod{2.3^n}$,

Then $x^3 \equiv (2.3^{n-2}k + 3)^3 \pmod{2.3^n}$

$\equiv (2.3^{n-2}k)^3 + 3.(2.3^{n-2}k)^2.3 + 3.2.3^{n-2}k.3^2 + 3^3 \pmod{2.3^n}$

$\equiv 2.3^n k\{3^2 + 2.3^{n-1}k.3 + 2^2.3^{2n-3}k\} + 3^3 \pmod{2.3^n}$

$\equiv 3^3 \pmod{2.3^n}$, for a for any positive integer.

Thus, $x \equiv 2.3^{n-2}k + 3 \pmod{2.3^n}$ satisfies the said congruence.
Hence, it is a solution of the congruence.
But for $k = 9$, the solution becomes $x \equiv 2.3^{n-2}.9 + 3 \pmod{2.3^n}$
$\equiv 2.3^n + 3 \pmod{2.3^n}$
$\equiv 3 \pmod{2.3^n}$
Which is the same as for the solution for $k = 0$.
For $k = 10, 11, \dots\dots\dots$ the solutions are also the same as for $k = 1, 2 \dots\dots$
Thus, the congruence has exactly nine incongruent solutions
$x \equiv 2.3^{n-2}k + 3 \pmod{2.3^n}$, for any integer a, $k = 0, 1, 2, 3, 4, 5, 6, 7, 8$.
But if $a = 3l, l = 2, 3, \dots\dots$, then for $x \equiv 2.3^{n-3}k + 3\,l \pmod{2.3^n}$,
$x^3 \equiv (2.3^{n-3}k + 3\,l\,)^3 \ (mod\ 2.3^n)$
$\equiv (2.3^{n-3}k)^3 + 3.(2.3^{n-3}k)^2.3l + 3.2.3^{n-3}k.(3l)^2 + (3\,l)^3 \ (mod\ 2.3^n)$
$\equiv (3\,l)^3 + 2.3^n k(\dots\dots) \ (mod 2.3^n)$
$\equiv (3\,l)^3 \ (mod 2.3^n)$.
Therefore, $x \equiv 2.3^{n-3}k + 3\,l \pmod{2.3^n}$ satisfies the cubic congruence and hence it is a solution of it.
But for $k = 27 = 3^3$, the solution reduces to $x \equiv 2.3^{n-3}.3^3 + 3\,l \pmod{2.3^n}$
$\equiv 2.3^n + 3l \ (mod\ 2.3^n)$
$\equiv 3l \ (mod\ 2.3^n)$
Thus, it is the same solution as for $k = 0$.
Similarly, it is also seen that for $k = 28, 29, \dots\dots$, the solutions repeat as for $k = 1, 2, \dots\dots$
Therefore, the congruence has twenty seven solutions: $x \equiv 2.3^{n-3}k + 3\,l \pmod{2.3^n}$;
$k =, 1, 2, \dots\dots\dots\dots, 26$.
Sometimes, in the cubic congruence, the integer a may not be a perfect cube. The readers have to make it so, by adding multiples of the modulus [1].

ILLUSTRATIONS
Consider the congruence: $x^3 \equiv 17 \pmod{54}$.
It can be written as $x^3 \equiv 17 + 2.54 = 125 = 5^3 \pmod{2.3^3}$ with $a = 5$.
It is of the type: $x^3 \equiv a^3 \pmod{2.3^3}$.
Such congruence has exactly three solutions which are giving by
$x \equiv 2.3^{n-1}k + a \pmod{2.3^n}; k = 0, 1, 2$.
$\equiv 2.3^2k + 5 \pmod{2.3^2}$. $\equiv 18k + 5 \pmod{54}$
$\equiv 3, 23, 41 \pmod{54}$; $k = 0, 1, 2$.
Consider the congruence: $x^3 \equiv 8 \pmod{2.3^3}$.
It can be written as $x^3 \equiv 8 = 2^3 \pmod{2.3^3}$ with $a = 2$.

Such congruence has exactly three solutions which are giving by
$x \equiv 2.3^{n-1}k + a \pmod{2.3^n}; k = 0, 1, 2$.
$\equiv 2.3^2k + 2 \pmod{2.3^3}; k = 0, 1, 2$.
$\equiv 18k + 2 \pmod{54}$
$\equiv 2, 20, 38 \pmod{54}$.
Consider the congruence: $x^3 \equiv 27 \pmod{54}$.
It can be written as $x^3 \equiv 27 = 3^3 \pmod{2.3^3}$ with $a = 3$.
It is of the type: $x^3 \equiv 3^3 \pmod{2.3^3}$.

Such congruence has exactly nine solutions which are giving by
$x \equiv 2.3^{n-2}k + 3 \pmod{2.3^n}; k = 0, 1, 2$.
$\equiv 2.3^1k + 3 \pmod{2.3^3}$. $\equiv 6k + 3 \pmod{54}$
$\equiv 3, 9, 15, 21, 27, 33, 39, 45, 51 \pmod{54}$; $k = 0, 1, 2, 3, 4, 5, 6, 7, 8$.
Consider the congruence: $x^3 \equiv 216 \pmod{486}$.
It can be written as $x^3 \equiv 216 = 6^3 \pmod{486}$ with $a = 6$.
It is of the type: $x^3 \equiv a^3 \pmod{2.3^n}$ $with\ n = 5$.

Such congruence has exactly twenty seven solutions which are giving by
$x \equiv 2.3^{n-3}k + 6 \pmod{2.3^n}; k = 0, 1, 2 \dots\dots\dots\dots\dots\dots\dots\dots, 26$.
$\equiv 2.3^2k + 6 \pmod{2.3^5}$. $\equiv 18k + 6 \pmod{486}$
$\equiv 6, 24, 42, 60, 78, 96, 114, 132, 150, 168, 186, 204, 222,$
$240, 258, 276, 296, 312, 330, 348, 366, 384, 402, 420, 438,$
$456, 474, \pmod{486}$; $k = 0, 1, 2, 3, \dots\dots\dots\dots\dots\dots\dots\dots, 26$.
Consider the congruence: $x^3 \equiv 729 \pmod{1458}$.
It can be written as $x^3 \equiv 729 = 9^3 \pmod{2.729}$ with $a = 9$.
$i.e. x^3 \equiv 9^3 \pmod{2.3^6}$

It is of the type: $x^3 \equiv a^3 \pmod{2.3^n}$ *with* $n = 6$.

Such congruence has exactly twenty seven solutions which are giving by
$x \equiv 2.3^{n-3}k + 9 \pmod{2.3^n}; k = 0, 1, 2 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .., 26.$
$\equiv 2.3^3k + 9 \pmod{2.3^3}. \quad \equiv 54k + 9 \pmod{1458}$
$\equiv 9, 63, 117, 171, 225, 279, 333, 387, 441, 495, 549, 603, 657, 711,$
$765, 819, 873, 927, 981, 1035, 1089, 1143, 1197, 1251, 1305, 1359,$
$1413 \pmod{1458}; \; k = 0, 1, 2, 3, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .., 26.$

CONCLUSION
Thus, it can be concluded that the said congruence: $x^3 \equiv a^3 \pmod{2.3^n}$ has exactly three incongruent solutions:

$$x \equiv 2.3^{n-1}k + a \pmod{2.3^n}; k = 0, 1, 2.$$

But if $a = 3l$, the congruence has nine solutions for $l = 1$:

$$x \equiv 2.3^{n-2}k + 3 \pmod{2.3^n}; k = 0, 1, 2 \ldots \ldots \ldots 8.$$

For $l = 2, 3, \ldots ..$,the congruence has twenty seven solutions:
$x \equiv 2.3^{n-3}k + 3 \pmod{2.3^n}; k = 0, 1, 2 \ldots \ldots \ldots 26.$

MERIT OF THE PAPER
In this paper, the standard cubic congruence of even composite modulus-an even multiple of power of three is studied for its solutions and is formulated. This lessens the labour of the readers to find the solutions. This is the merit of the paper.

REFERENCES

[1] Thomas Koshy, "Elementary Number Theory with Applications", 2/e (Indian print, 2009), Academic Press.

[2] Niven I., Zuckerman H. S., Montgomery H. L. (1960, Reprint 2008), "An Introduction to The Theory of Numbers", 5/e, Wiley India (Pvt) Ltd.

[3] Roy B M, "Discrete Mathematics & Number Theory", 1/e, Jan. 2016, Das Ganu Prakashan, Nagpur.

[4] Roy B M, Formulation of two special classes of standard cubic congruence of composite modulus- a power of Three, International Journal of Scientific Research and Engineering Development (IJSRED), ISSN: 2581-7175, Vol-02, Issue-03, May-June 2019, Page-288-291.

[5] Roy B M, Solving some cubic congruence of prime modulus, International Journal of Trend in Scientific Research and Development (IJTSRD), ISSN: 2456-6470, Vol-03, Issue-04, Jun-19.

[6] Roy B M, Formulation of a class of solvable standard cubic congruence of even composite modulus, International Journal of Advanced Research, Ideas, and Innovations in Technology (IJARIIT), ISSN: 2454-132X, Vol-05, Issue-01, Jan-Feb-19.