Serial Queues Connected To Non-Serial Channels With Feedback In Serial Channels And Reneging And Balking In Both Types Of Channels For Finite Waiting Space

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Abstract

Steady state solution of general queuing model having feedback, balking and reneging in serial queuing processes connected with non-serial queuing channels with reneging and balking with finite waiting space in random order selection for service has been studied in the present paper.

1. Introduction

Various researchers including O'Brien (1954), Barrer (1955) and Finch (1959) studied the problems of serial queues in steady-state with Poisson assumption. Singh (1984) studied the problem of serial queues introducing the concept of reneging. Singh and Singh (1994) worked on the network of queuing processes with impatient customers. Punam, et.al (2011) found the steady-state solution of serial queuing processes where feedback is not permitted. Satyabir, et.al (2014) obtained steady-state solution of serial queues with feedback, balking and reneging. However there may be situations where the serial queuing processes may be connected with non-serial queuing channels keeping the above observations in view, we in the present paper obtained the steady-state solutions for serial queuing processes with feedback, balking and reneging connected with non-serial queuing channels with reneging and balking in which

(i) M-serial queuing processes with feedback, balking and reneging connected with N-non-serial queuing channels with reneging and balking.

(ii) A customer may join any channel from outside and leave the system at any stage after getting service.

(iii) Feedback is permitted from each channel to its previous channel in serial channels.

(iv) The customer may balk due to long queue at each serial and non-serial service channel.

(v) The impatient customer leaves both serial and non serial service channels after wait of certain time.

(vi) The input process in serial and non-serial channels depends upon queue size and Poisson arrivals are followed. Exponential service times are followed.

(vii) The queue discipline is random selection for service. Waiting space is finite.

Key Words : Steady-State, difference-differential, waiting space, random selection, Poisson arrivals, exponential service, feedback, balking and reneging.

2. Formulation of the Model

The system consists of the serial queues $Q_j\ (j=1,2,3,....,M)$ and non-serial channels $Q_{1i}\ (i=1,2,3,....N)$ with respective servers $S_j\ (j=1,2,3,....,M)$ and $S_{1i}\ (i=1,2,3,....N)$. Customers demanding different types of service arrive from outside the system in Poisson stream with parameters $\lambda_j\ (j=1,2,3,......,M)$ and $\lambda_{1i}\ (i=1,2,3,...N)$ at $Q_j\ (j=1,2,3,....,M)$ and $Q_{1i}\ (i=1,2,3,....N)$ but the sight of long queue at $Q_j\ (\mathrm{j}=1,2,3,....\mathrm{M})$ and $Q_{1i}\ (i=1,2,3,....N)$ may discourage the fresh customer from joining it and may decide not to enter the service channel at $Q_j\ (\mathrm{j}=1,2,3,....\mathrm{M})$ and $Q_{1i}\ (i=1,2,3,....N)$. Then the Poisson input rate at Q_j (j=1,2,3,....M) would be $\frac{\lambda_j}{n_j+1}$ where n_j is the queue size of Q_j (j=1,2,3,....M) and $\frac{\lambda_{1i}}{m_i+1}$ where m_i is the queue size of $Q_{1i}\,(i=1,2,3,....N)$. Further, the impatient customer joining any serial service channel $Q_j\,(j=1,2,3,....,M)$ and non-serial channel $Q_{1i}\,(i=1,2,3,....N)$ may leave the queue without getting service after wait of certain time. Here C_{in_i} and D_{jm_j} are reneging rates at which customer renege after a wait of time T_{0i} whenever there are n_i and m_j customer in the service channels Q_i and Q_{1j}.

$C_{in_i}=\dfrac{\mu_{1i}e^{-\frac{\mu_{1i}T_{0i}}{n_i}}}{1-e^{-\frac{\mu_{1i}T_{0i}}{n_i}}}$ $(i=1,2,3,....M)$ and $D_{jm_j}=\dfrac{\mu_{1j}e^{-\frac{\mu_{1j}T_{0j}}{m_j}}}{1-e^{-\frac{\mu_{1j}T_{0j}}{m_j}}}$ $(j=1,2,3,....N)$. Service time distributions for servers $S_j\,(j=1,2,3,....,M)$ and $S_{1i}\,(i=1,2,3,....N)$ are mutually independent negative exponential distribution with parameters $\mu_j\,(j=1,2,......,M)$ and $\mu_{1i}\,(i=1,2,3,...N)$ respectively. After the completion of service at S_j, the customer either leaves the system with probability p_j or joins the next channel with probability $\frac{q_j}{n_{j+1}+1}$ or join back the previous channel with probability $\frac{r_j}{n_{j-1}+1}$ such that $p_j+\frac{q_j}{n_{j+1}+1}+\frac{r_j}{n_{j-1}+1}=1$ $(j=1,2,3,...M-1)$ and after the completion of service at S_M the customer either leaves the system with probability p_M or join back the previous channel with probability $\frac{r_M}{n_{M-1}+1}$ or join any queue $Q_{1i}\,(i=1,2,3,.....N)$ with probability $\frac{q_{Mi}}{m_i+1}$ $(i=1,2,3,....N)$ such that $p_M+\frac{r_M}{n_{M-1}+1}+\sum_{i=1}^{N}\frac{q_{Mi}}{m_i+1}=1$. It is being mentioned here that $r_j=0$ for $j=1$ as there is no previous channel of the first channel.

3. Formulation of Equations

Here we assume that at any instant there are K customers in the system i.e. $\sum_{i=1}^{M}n_i+\sum_{j=1}^{N}m_j=K$.

Then the customers arriving at that instant will not be allowed to join the system and is considered lost for the system.

Define $P(\tilde{n},\tilde{m};t)$ and operators $T_i.$, $T._i$, $T._{i,i+1}.$, $T_{i-1}.,._i$ to act upon the vectors $\tilde{n}=(n_1,n_2,n_3,.....n_M)$ or $\tilde{m}=(m_1,m_2,m_3,.....m_N)$ as in model A.

Following the procedure given by Kelly (1979), we write the difference – differential equations as

$$\frac{d\,P(\tilde{n},\tilde{m};t)}{dt}=-\left[\sum_{i=1}^{M}\frac{\lambda_i}{n_i+1}+\sum_{i=1}^{M}\delta(n_i)(\mu_i+C_{in_i})+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j+1}+\sum_{j=1}^{N}\delta(m_j)(\mu_{1j}+D_{jm_j})\right]P(\tilde{n},\tilde{m};t)+\sum_{i=1}^{M}\frac{\lambda_i}{n_i}P(T_i.(\tilde{n}),\tilde{m};t)$$

$$+\sum_{i=1}^{M}(\mu_i p_i+C_{in_i+1})P(T._i(\tilde{n}),\tilde{m};t)+\sum_{i=1}^{M-1}\mu_i\frac{q_i}{n_{i+1}}P(T._{i,i+1}.(\tilde{n}),\tilde{m};t)+\sum_{i=1}^{M}\mu_i\frac{r_i}{n_{i-1}}P(T_{i-1}.,._i(\tilde{n}),\tilde{m};t).$$

$$+\sum_{j=1}^{N}\mu_M\frac{q_{Mj}}{m_j}P(n_1,n_2,....n_M+1,T_j.(\tilde{m});t)+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P(\tilde{n},T_j.(\tilde{m});t)+\sum_{j=1}^{N}(\mu_{1j}+D_{jm_j+1})P(\tilde{n},T._j(\tilde{m});t)\quad \text{for}$$

$n_i\ge 0$ $(i=1,2,3,....,M)$, $m_j\ge 0\,(j=1,2,3,...N)$; and $\sum_{i=1}^{M}n_i+\sum_{j=1}^{N}m_j<K$.

$$\frac{d\,P(\tilde{n},\tilde{m};t)}{dt}=-\left[\sum_{i=1}^{M}\delta(n_i)(\mu_i+C_{in_i})+\sum_{j=1}^{N}\delta(m_j)(\mu_{1j}+D_{jm_j})\right]P(\tilde{n},\tilde{m};t)+\sum_{i=1}^{M}\frac{\lambda_i}{n_i}P(T_i.(\tilde{n}),\tilde{m};t)+\sum_{i=1}^{M-1}\mu_i\frac{q_i}{n_{i+1}}P(T._{i,i+1}.(\tilde{n}),\tilde{m};t)$$

$$+\sum_{i=1}^{M}\mu_i\frac{r_i}{n_{i-1}}P(T_{i-1}.,._i(\tilde{n}),\tilde{m};t).$$

$$+\sum_{j=1}^{N}\mu_M \frac{q_{Mj}}{m_j}P\left(n_1,n_2,\ldots.n_M+1,T_j.(\tilde{m});t\right)+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P\left(\tilde{n},T_j.(\tilde{m});t\right) \qquad (3.2)$$

for $n_i \geq 0 \quad (i=1,2,3,\ldots.,M)$, $m_j \geq 0 \, (j=1,2,3,\ldots N)$;

where $\delta(x)=\begin{cases}1 & if \quad x\neq 0\\ 0 & if \quad x=0\end{cases}$ and $\sum_{i=1}^{M}n_i+\sum_{j=1}^{N}m_j=K$.

4. Steady-State Equations

We write the following steady-state equations of the queuing model by equating the time derivative to zero in equations (3.1) and (3.2).

$$\left[\sum_{i=1}^{M}\frac{\lambda_i}{n_i+1}+\sum_{i=1}^{M}\delta(n_i)\left(\mu_i+C_{in_i}\right)+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j+1}+\sum_{j=1}^{N}\delta(m_j)\left(\mu_{1j}+D_{jm_j}\right)\right]P(\tilde{n},\tilde{m})=\sum_{i=1}^{M}\frac{\lambda_i}{n_i}P\left(T_i.(\tilde{n}),\tilde{m}\right)$$

$$+\sum_{i=1}^{M}\left(\mu_i p_i+C_{in_i+1}\right)P\left(T._i(\tilde{n}),\tilde{m}\right)+\sum_{i=1}^{M-1}\mu_i\frac{q_i}{n_{i+1}}P\left(T._{i},_{i+1}.(\tilde{n}),\tilde{m}\right)+\sum_{i=1}^{M}\mu_i\frac{r_i}{n_{i-1}}P\left(T_{i-1}.,._i(\tilde{n}),\tilde{m}\right).$$

$$+\sum_{j=1}^{N}\mu_M\frac{q_{Mj}}{m_j}P\left(n_1,n_2,\ldots.n_M+1,T_j.(\tilde{m})\right)+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P\left(\tilde{n},T_j.(\tilde{m})\right)+\sum_{j=1}^{N}\left(\mu_{1j}+D_{jm_j+1}\right)P\left(\tilde{n},T._j(\tilde{m})\right)$$

(4.1)

$$\left[\sum_{i=1}^{M}\delta(n_i)\left(\mu_i+C_{in_i}\right)+\sum_{j=1}^{N}\delta(m_j)\left(\mu_{1j}+D_{jm_j}\right)\right]P(\tilde{n},\tilde{m})$$

$$=\sum_{i=1}^{M}\frac{\lambda_i}{n_i}P\left(T_i.(\tilde{n}),\tilde{m}\right)+\sum_{i=1}^{M-1}\mu_i\frac{q_i}{n_{i+1}}P\left(T._{i},_{i+1}.(\tilde{n}),\tilde{m}\right)+\sum_{i=1}^{M}\mu_i\frac{r_i}{n_{i-1}}P\left(T_{i-1}.,._i(\tilde{n}),\tilde{m}\right).$$

$$+\sum_{j=1}^{N}\mu_M\frac{q_{Mj}}{m_j}P\left(n_1,n_2,\ldots.n_M+1,T_j.(\tilde{m})\right)+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j}P\left(\tilde{n},T_j.(\tilde{m})\right) \qquad (4.2)$$

where $\delta(x)=\begin{cases}1 & if \quad x\neq 0\\ 0 & if \quad x=0\end{cases}$ and $\sum_{i=1}^{M}n_i+\sum_{j=1}^{N}m_j=K$.

5. Steady-State Solutions

The steady state solutions of equations (4.1) and (4.2) can be verified as

$$P(\tilde{n},\tilde{m})=P(\tilde{0},\tilde{0})\left(\frac{1}{n_1!}\frac{\left(\lambda_1+\frac{\mu_2 r_2\rho_2}{(n_2+1)\left(\mu_2+C_{2n_2+1}\right)}\right)^{n_1}}{\prod_{i=1}^{n_1}\left(\mu_1+C_{1i}\right)}\right)\left(\frac{1}{n_2!}\frac{\left(\lambda_2+\frac{\mu_1 q_1\rho_1}{(n_1+1)\left(\mu_1+C_{1n_1+1}\right)}+\frac{\mu_3 r_3\rho_3}{(n_3+1)\left(\mu_3+C_{3n_3+1}\right)}\right)^{n_2}}{\prod_{i=1}^{n_2}\left(\mu_2+C_{2i}\right)}\right).$$

$$.\left(\frac{1}{n_3!}\frac{\left(\lambda_3+\frac{\mu_2 q_2\rho_2}{(n_2+1)\left(\mu_2+C_{2n_2+1}\right)}+\frac{\mu_4 r_4\rho_4}{(n_4+1)\left(\mu_4+C_{4n_4+1}\right)}\right)^{n_3}}{\prod_{i=1}^{n_3}\left(\mu_3+C_{3i}\right)}\right)\ldots$$

$$\cdot\left(\frac{1}{n_{M-1}!}\frac{\left(\lambda_{M-1}+\frac{\mu_{M-2}q_{M-2}\rho_{M-2}}{(n_{M-2}+1)\left(\mu_{M-2}+C_{M-2n_{M-2}+1}\right)}+\frac{\mu_M r_M \rho_M}{(n_M+1)\left(\mu_M+C_{Mn_M+1}\right)}\right)^{n_{M-1}}}{\prod_{i=1}^{n_{M-1}}(\mu_{M-1}+C_{M-1i})}\right).$$

$$\cdot\left(\frac{1}{n_M!}\frac{\left(\lambda_M+\frac{\mu_{M-1}q_{M-1}\rho_{M-1}}{(n_{M-1}+1)\left(\mu_{M-1}+C_{M-1n_{M-1}+1}\right)}\right)^{n_M}}{\prod_{i=1}^{n_M}(\mu_M+C_{Mi})}\right)\left(\frac{1}{m_1!}\frac{\left(\lambda_{11}+\frac{\mu_M q_{M1}\rho_M}{(n_M+1)(\mu_M+C_{Mn_M+1})}\right)^{m_1}}{\prod_{j=i}^{m_1}(\mu_{11}+D_{1j})}\right).$$

$$\cdot\left(\frac{1}{m_2!}\frac{\left(\lambda_{12}+\frac{\mu_M q_{M2}\rho_M}{(n_M+1)(\mu_M+C_{Mn_M+1})}\right)^{m_2}}{\prod_{j=i}^{m_2}(\mu_{12}+D_{2j})}\right)\cdots\cdots\left(\frac{1}{m_N!}\frac{\left(\lambda_{1N}+\frac{\mu_M q_{MN}\rho_M}{(n_M+1)(\mu_M+C_{Mn_M+1})}\right)^{m_N}}{\prod_{j=i}^{m_1}(\mu_{1N}+D_{Nj})}\right) \quad (5.1)$$

with relation $\sum_{i=1}^{M} n_i + \sum_{j=1}^{N} m_j \le K$, $n_i \ge 0 \quad (i = 1, 2, 3, ..., M)$, $m_j \ge 0\,(j = 1, 2, 3, ...N)$.

Where $\rho_1 = \lambda_1 + \frac{\mu_2 r_2 \rho_2}{(n_2+1)\left(\mu_2+C_{2n_2+1}\right)}$

$$\rho_2 = \left(\lambda_2 + \frac{\mu_1 q_1 \rho_1}{(n_1+1)\left(\mu_1+C_{1n_1+1}\right)} + \frac{\mu_3 r_3 \rho_3}{(n_3+1)\left(\mu_3+C_{3n_3+1}\right)}\right)$$

$$\rho_3 = \lambda_3 + \frac{\mu_2 q_2 \rho_2}{(n_2+1)\left(\mu_2+C_{2n_2+1}\right)} + \frac{\mu_4 r_4 \rho_4}{(n_4+1)\left(\mu_4+C_{4n_4+1}\right)} \quad (5.2)$$

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$$\rho_{M-1} = \lambda_{M-1} + \frac{\mu_{M-2} q_{M-2} \rho_{M-2}}{(n_{M-2}+1)\left(\mu_{M-2}+C_{M-2n_{M-2}+1}\right)} + \frac{\mu_M r_M \rho_M}{(n_M+1)\left(\mu_M+C_{Mn_M+1}\right)}$$

$$\rho_M = \lambda_M + \frac{\mu_{M-1} q_{M-1} \rho_{M-1}}{(n_{M-1}+1)\left(\mu_{M-1}+C_{M-1n_{M-1}+1}\right)}$$

Solving these (5.2) M-equations for ρ_M with the help of determinants, we get

$$\rho_M = \frac{\left(\begin{array}{l} \lambda_M \Delta_{M-1} + \frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)(\mu_{M-1}+C_{M-1n_{M-1}+1})}\lambda_{M-1}\Delta_{M-2} + \\ \frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)(\mu_{M-1}+C_{M-1n_{M-1}+1})}\cdot\frac{q_{M-2}\mu_{M-2}}{(n_{M-2}+1)(\mu_{M-2}+C_{M-2n_{M-2}+1})}\lambda_{M-2}\Delta_{M-3} + \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ +\frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)(\mu_{M-1}+C_{M-1n_{M-1}+1})}\cdot\frac{q_{M-2}\mu_{M-2}}{(n_{M-2}+1)(\mu_{M-2}+C_{M-2n_{M-2}+1})}\ldots \\ \cdot\frac{q_3\mu_3}{(n_3+1)(\mu_3+C_{3n_3+1})}\lambda_3\Delta_2 \\ +\frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)(\mu_{M-1}+C_{M-1n_{M-1}+1})}\cdot\frac{q_{M-2}\mu_{M-2}}{(n_{M-2}+1)(\mu_{M-2}+C_{M-2n_{M-2}+1})}\ldots \\ \cdot\frac{q_3\mu_3}{(n_3+1)(\mu_3+C_{3n_3+1})}\cdot\frac{q_2\mu_2}{(n_2+1)(\mu_2+C_{2n_2+1})}\lambda_2\Delta_1 \\ +\frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)(\mu_{M-1}+C_{M-1n_{M-1}+1})}\cdot\frac{q_{M-2}\mu_{M-2}}{(n_{M-2}+1)(\mu_{M-2}+C_{M-2n_{M-2}+1})}\ldots \\ \cdot\frac{q_3\mu_3}{(n_3+1)(\mu_3+C_{3n_3+1})}\cdot\frac{q_2\mu_2}{(n_2+1)(\mu_2+C_{2n_2+1})}\cdot\frac{q_1\mu_1}{(n_1+1)(\mu_1+C_{1n_1+1})}\lambda_1 \end{array}\right)}{\left(\Delta_{M-1} - \frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)(\mu_{M-1}+C_{M-1n_{M-1}+1})}\cdot\frac{r_M\mu_M}{(n_M+1)(\mu_M+C_{Mn_M+1})}\Delta_{M-2}\right)} \quad (5.3)$$

Where $\Delta_M = \Delta_{M-1} - \frac{q_{M-1}\mu_{M-1}}{(n_{M-1}+1)(\mu_{M-1}+C_{M-1n_{M-1}+1})}\cdot\frac{r_M\mu_M}{(n_M+1)(\mu_M+C_{Mn_M+1})}\Delta_{M-2}$

Where

$$\Delta_1 = 1 \quad \Delta_2 = \begin{vmatrix} 1 & -\frac{\frac{r_2\mu_2}{n_2+1}}{\mu_2+c_{2n_2+1}} \\ -\frac{\frac{q_1\mu_1}{n_1+1}}{\mu_1+c_{1n_1+1}} & 1 \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} 1 & -\frac{\frac{r_2\mu_2}{n_2+1}}{\mu_2+c_{2n_2+1}} & 0 \\ -\frac{\frac{q_1\mu_1}{n_1+1}}{\mu_1+c_{1n_1+1}} & 1 & -\frac{\frac{r_3\mu_3}{n_3+1}}{\mu_3+c_{3n_3+1}} \\ 0 & -\frac{\frac{q_2\mu_2}{n_2+1}}{\mu_2+c_{2n_2+1}} & 1 \end{vmatrix}$$

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$$\Delta_M = \begin{vmatrix} 1 & -\dfrac{\frac{r_2}{n_2+1}\mu_2}{\mu_2+C_{2n_2+1}} & 0 & 0 & - & - & - & 0 & 0 & 0 \\ -\dfrac{\frac{q_1}{n_1+1}\mu_1}{\mu_1+C_{1n_1+1}} & 1 & -\dfrac{\frac{r_3}{n_3+1}\mu_3}{\mu_3+C_{3n_3+1}} & 0 & - & - & - & 0 & 0 & 0 \\ 0 & -\dfrac{\frac{q_2}{n_2+1}\mu_2}{\mu_2+C_{2n_2+1}} & 1 & -\dfrac{\frac{r_4}{n_4+1}\mu_4}{\mu_4+C_{4n_4+1}} & - & - & - & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & - & - & - & -\dfrac{\frac{q_{M-2}}{n_{M-2}+1}\mu_{M-2}}{\mu_{M-2}+C_{M-2n_{M-2}+1}} & 1 & -\dfrac{\frac{r_M}{n_M+1}\mu_M}{\mu_M+C_{Mn_M+1}} \\ 0 & 0 & 0 & 0 & - & - & - & 0 & -\dfrac{\frac{q_{M-1}}{n_{M-1}+1}\mu_{M-1}}{\mu_{M-1}+C_{M-1n_{M-1}+1}} & 1 \end{vmatrix}$$

Since ρ_M is obtained, so we can get ρ_{M-1} by putting the value of ρ_M in the last equation of (5.2), ρ_{M-2} by putting the values of ρ_{M-1} and ρ_M in the last but one equation of (5.2). Continuing in this way, we shall obtain $\rho_{M-3}, \rho_{M-4}, ---, \rho_3, \rho_2$ and ρ_1.

Thus, we write (5.1) as under

$$p(\tilde{n},\tilde{m}) = P(\tilde{0},\tilde{0})\left(\frac{1}{n_1!}\frac{(\rho_1)^{n_1}}{\prod_{i=1}^{n_1}(\mu_1+C_{1i})}\right)\left(\frac{1}{n_2!}\frac{(\rho_2)^{n_2}}{\prod_{i=1}^{n_2}(\mu_2+C_{2i})}\right)\left(\frac{1}{n_3!}\frac{(\rho_3)^{n_3}}{\prod_{i=1}^{n_3}(\mu_3+C_{3i})}\right)\cdots$$

$$\cdot\left(\frac{1}{n_{M-1}!}\frac{(\rho_{M-1})^{n_{M-1}}}{\prod_{i=1}^{n_{M-1}}(\mu_{M-1}+C_{M-1i})}\right)\cdot\left(\frac{1}{n_M!}\frac{(\rho_M)^{n_M}}{\prod_{i=1}^{n_M}(\mu_M+C_{Mi})}\right).$$

$$\cdot \quad \cdot\left(\frac{1}{m_1!}\frac{\left(\lambda_{11}+\frac{\mu_M q_{M1}\rho_M}{(n_M+1)(\mu_M+C_{Mn_M+1})}\right)^{m_1}}{\prod_{j=i}^{m_1}(\mu_{11}+D_{1j})}\right)\left(\frac{1}{m_2!}\frac{\left(\lambda_{12}+\frac{\mu_M q_{M2}\rho_M}{(n_M+1)(\mu_M+C_{Mn_M+1})}\right)^{m_2}}{\prod_{j=i}^{m_2}(\mu_{12}+D_{2j})}\right).$$

$$\cdots\left(\frac{1}{m_N!}\frac{\left(\lambda_{1N}+\frac{\mu_M q_{MN}\rho_M}{(n_M+1)(\mu_M+C_{Mn_M+1})}\right)^{m_N}}{\prod_{j=i}^{m_1}(\mu_{1N}+D_{Nj})}\right) \qquad (5.5)$$

$n_i \geq 0 \quad (i = 1,2,3,....,M)$, $m_j \geq 0\,(j = 1,2,3,...N)$ We obtain $P(\tilde{0},\tilde{0})$ from the normalizing conditions.

$$\sum_{\tilde{n}=\tilde{0},\tilde{m}=\tilde{0}}^{\infty} P(\tilde{n},\tilde{m}) = 1 \qquad (5.6)$$

and with the restriction that traffic intensity of each service channel of the system is less than unity.

Here it is mentioned that the customers leave the system at constant rate as long as there is a line, provided that the customers are served in the order in which they arrive. Putting $C_{in_i} = C_i \quad (i = 1, 2, 3, ... M)$ and $D_{jm_j} = D_j \quad (j = 1, 2, 3, ... N)$ in the steady-state solution (5.1) then $\rho_i \ (i = 1, 2, 3, ... M)$ will change accordingly and the steady- state solution reduces to

$$P(\tilde{n}, \tilde{m}) = P(\tilde{0}, \tilde{0}) \left(\frac{1}{n_1!} \left(\frac{\rho_1}{\mu_1 + C_1} \right)^{n_1} \right) \left(\frac{1}{n_2!} \left(\frac{\rho_2}{\mu_2 + C_2} \right)^{n_2} \right) \left(\frac{1}{n_3!} \left(\frac{\rho_3}{\mu_3 + C_3} \right)^{n_3} \right) ....$$

$$. \left(\frac{1}{n_{M-1}!} \left(\frac{\rho_{M-1}}{\mu_{M-1} + C_{M-1}} \right)^{n_{M-1}} \right) \left(\frac{1}{n_M!} \left(\frac{\rho_M}{\mu_M + C_M} \right)^{n_M} \right) . \left(\frac{1}{m_1!} \left(\frac{\rho_{11}}{(\mu_{11} + D_1)} \right)^{m_1} \right) . \qquad (5.7)$$

$$. \left(\frac{1}{m_2!} \left(\frac{\rho_{12}}{(\mu_{12} + D_2)} \right)^{m_2} \right) ... \left(\frac{1}{m_N!} \left(\frac{\rho_{1N}}{(\mu_{1N} + D_N)} \right)^{m_N} \right)$$

where $\rho_{1j} = \lambda_{1j} + \dfrac{\mu_M q_{Mj} \rho_M}{(n_M + 1)(\mu_M + C_M)}, \qquad j = 1, 2, 3, ... N$

We obtain $P(\tilde{0}, \tilde{0})$ from (5.6) and (5.7) as

$$\left(P(\tilde{0}, \tilde{0}) \right)^{-1} = \prod_{i=1}^{M} e^{\frac{\rho_i}{\mu_i + C_i}} \prod_{j=1}^{N} e^{\frac{\rho_{1j}}{\mu_{1j} + D_j}}$$

Thus $P(\tilde{n}, \tilde{m})$ is completely determined.

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