Fundamental solution for generalized thermoelastic media with diffusion

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ABSTRACT: Thermo-diffusion in the solids is one of the transport processes that has major practical importance and occur due to coupling of the fields of temperature, mass diffusion and that of strain. This process is used in thermo-mechanical treatment of metals like carbonizing, nitriding steel etc. In the present problem, the system of differential equations of motion in the theory of nonlocal homogenous isotropic thermoelastic media with diffusion is investigated and established the fundamental solution of differential equations in case of steady oscillations. Differential vector calculus is used for obtaining the fundamental solution and some basic properties. Particular case is also deduced from the present work and compared with the existing result. The results of particular case exactly match with the existing results which justify the correctness of present work. This analysis of fundamental solution is very helpful to investigate two dimensional as well as three dimensional problems of nonlocal homogenous isotropic elastic solid with diffusion. Also, this study is very useful in different fields of geophysics and electronics.

KEYWORDS: Diffusion, fundamental solution, steady oscillations, nonlocal, thermoelasticity.

1. INTRODUCTION

Generalized dynamical theory of thermoelasticity was developed by Lord and Shulman [1] using time dependent heat transport equation. With the help of entropy production inequality in classical thermoelasticity, Green and Lindsay [2] discussed the restrictions on constitutive equations. Gurtin [3] studied the temperature distribution due to action of internal forces. The linear theory of elasticity explains the effective properties of various materials like wood, concrete and steel etc but is unable to describe the properties of nano structure. The theory of nonlocal elasticity takes account of action between nano particles by considering the fact that stresses at a point not only depend on strain at that point but also on all points of the body. Eringrn [4-6] extended the theory of elasticity to theory of nonlocal elasticity. Green and Naghdi [7] introduced a new thermodynamical theory by using general entropy balance and discussed the thermoelasticity without energy dissipation. Nowacki [8-9] established the asymptotic solution to boundary value problem of three dimensional micropolar theory of elasticity. Kumar and Kumar [10] investigated the plane wave propagation in nonlocal micropolar thermoelastic media. Kaur and Singh [11] studied plane wave propagation in a nonlocal magneto-thermoelastic semiconductor solid with rotation and three phase lag fractional order heat transfer.

Diffusion is the spontaneous movement of particles from a region of higher concentration to region of lower concentration and thermal diffusion takes the account of heat transfer. Auoadi [12-14] derived the constitutive equations and equation of motion for a generalized thermoelastic diffusion with one relaxation time and proved uniqueness theorem for these equations by using Laplace transforms. H$\ddot{o}$rmander [15-16] analyzed the partial differential operators which are of great importance in order to find the fundamental solution in the thermoelastic diffusion solid. To study the boundary value problem of thermoelasticity, it is required to evaluate the fundamental solution of the system of partial differential equation. Fundamental solution in the classical theory of coupled thermoelasticity was firstly discussed by Hetnarski [17-18]. Svanadze [19-22] established the fundamental solution of differential equations of steady oscillations in different types of thermoelastic solids. Scarpetta [23] constructed fundamental solution in the theory of micropolar elasticity. elasticity. Fundamental solution in the theory of thermoelastic diffusion was formulated by Kumar and Kansal [24-25]. Many problems related to plane wave propagation and fundamental solution have been discussed by some of other researchers like Sharma and Kumar [26], Kumar et al. [27], Kumar and Devi [28], Biswas [29], Kumar and Batra [30], Poonam et.al [31], Kumar et al. [32], Kumar and Batra [33]. However, from the best of author's knowledge, no study has been done for investigating the combining effect of nonlocal and diffusion on fundamental solution of generalized thermoelastic solid. In current problem, we established the fundamental solution of differential equations in case of steady oscillations in terms of elementary functions for nonlocal thermoelastic solid with diffusion. Some basic properties and special case are also studied.

2. MATHEMATICAL FORMULATION

Let $\boldsymbol{X} = (x, y, z)$ be a point in the three dimensional Euclidean space E^3, $\boldsymbol{D}_x \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ and t represents the time variable. Following Eringen [4-6], governing equations and the constitutive relations for nonlocal generalised thermoelastic solid with diffusion are given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1}$$

$$(1 - \varepsilon'^2\nabla^2)\sigma_{ij} = \hat{\sigma}_{ij} = 2\mu e_{ij} + [\lambda e_{pp} - \beta_1' T' - \beta_2' C']\delta_{ij} \tag{2}$$

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ij} - \beta_1' T'_{,i} - \beta_2' C'_{,i} = (1 - \varepsilon'^2\nabla^2)\ddot{u}_i \tag{3}$$

$$\rho C_e(\dot{T}' + \eta_0 \ddot{T}') + \beta_1' T_0(\dot{e}_{pp} + \eta_0 \ddot{e}_{pp}) + a' T_0(\dot{C}' + \eta_0 \ddot{C}') = K T'_{,ii} \quad (4)$$
$$D' \beta_2' e_{pp,ii} + D' a' T'_{,ii} + (1 - \varepsilon'^2 \nabla^2)(\dot{C}' + \eta \ddot{C}') = D' b' C'_{,ii} \quad (5)$$

where $\boldsymbol{u} = (u_1, u_2, u_3)$ is the displacement vector; e_{ij} are components of strain tensor and σ_{ij} are the stress components; e_{pp} is dilatation; $\hat{\sigma}_{ij}$ $(i, j = 1,2,3)$ denotes the strain components corresponding to the local thermoelastic solid with diffusion; T' is the temperature change measured from the absolute temperature T_0; K is the thermal conductivity and C_e is the specific heat; ρ is density of the medium; η_0 denotes the thermal relaxation time; C' is the concentration; η is the relaxation time of diffusion; D' is the thermoelastic diffusion constant; a' and b' respectively measure the thermo-diffusion effects and diffusive effects; β_1', β_2' are material coefficients with $\beta_1' = (3\lambda + 2\mu)\alpha_t$, $\beta_2' = (3\lambda + 2\mu)\alpha_c$; α_t the coefficient of linear thermal expansion and α_c is the coefficient of linear diffusion expansion; λ and μ are Lame's constants; ∇^2 denotes the Laplacian operator; $\varepsilon' = e_0 a$ is the non-local parameter; e_0 corresponds to the material constant; a denotes the characteristic length; δ_{ij} is Kronecker delta. In the above equations, superposed dot represents the derivative with respect to time and ‘,’ in the subscript denotes the partial derivatives with respect to x, y, z for $i, j = 1,2,3$ respectively.

For two-dimensional problem, we assume that all quantities related to the medium are functions of cartesian coordinates x, z and time t and are independent of y $\left(i.e. \frac{\partial}{\partial y} \equiv 0\right)$. we take displacement vector as

$$\boldsymbol{u} = (u_1, 0, u_3) \quad (6)$$

We define dimensionless quantities as

$$u_1' = \frac{\rho \omega' c_0}{\beta_1' T_0} u_1,\ u_3' = \frac{\rho \omega' c_0}{\beta_1' T_0} u_3,\ x' = \frac{\omega' x}{c_0},\quad z' = \frac{\omega' z}{c_0}, t' = \omega' t$$
$$T' = \frac{T'}{T_0}, C' = \frac{\beta'_2}{\beta_1' T_0} C', \eta_0' = \omega' \eta_0, \eta' = \omega' \eta \quad (7)$$

where $\omega' = \frac{\rho C_e c_0^2}{K}$, $c_0 = \sqrt{\frac{\lambda + 2\mu}{\rho}}$

Using equation 7 in equations 3 − 5 and suppressing the primes, we get

$$\chi_1 \nabla^2 \boldsymbol{u} + \chi_2\, grad\ div\ \boldsymbol{u} - grad\ T' - grad\ C' = (1 - \varepsilon_1^2 \nabla^2)\ddot{\boldsymbol{u}} \quad (8)$$
$$\eta_t^{0\prime}[\dot{T}' + \chi_3\, div\ \dot{\boldsymbol{u}} + \chi_4 \dot{C}'] = \nabla^2 T' \quad (9)$$
$$\chi_5 \nabla^2\, div\ \boldsymbol{u} + \chi_6 \nabla^2 T' - \chi_7 \nabla^2 C' + (1 - \varepsilon_1^2 \nabla^2)\eta_c^{0\prime} \dot{C}' = 0 \quad (10)$$

where

$$\chi_1 = \frac{\lambda + \mu}{\lambda + 2\mu}, \chi_2 = \frac{\mu}{\lambda + 2\mu}, \chi_3 = \frac{\beta_1'^2 T_0}{\rho K \omega'}\ \chi_4 = \frac{a' \beta_1' T_0 c_0^2}{K \omega' \beta_2'}, \chi_5 = \frac{D' \beta_2'^2 \omega'}{\rho c_0^4},$$
$$\chi_6 = \frac{D' a' \omega' \beta_2'}{\beta_1' c_0^2}, \chi_7 = \frac{D' b' \omega'}{c_0^2}, \varepsilon_1^2 = \frac{\varepsilon'^2 \omega'^2}{c_0^2}, \eta_t^{0\prime} = 1 + \eta_0 \omega' \frac{\partial}{\partial t}, \eta_c^{0\prime} = 1 + \eta \omega' \frac{\partial}{\partial t}$$

We assume that displacement vector, temperature change and concentration are functions as

$$\left(\boldsymbol{u}(x,z,t), T'(x,z,t), C'(x,z,t)\right) = Re\left[\left(\boldsymbol{u}^*(x,z,t), T^*(x,z,t), C^*(x,z,t)\right)e^{-i\omega t}\right] \quad (11)$$

Using equation 11 in equations 8 − 10, we obtain following system of equations of steady oscillations

$$[(\chi_1 - \omega^2 \varepsilon_1^2)\nabla^2 + \omega^2]\boldsymbol{u}^* + \chi_2\, grad\ div\ \boldsymbol{u}^* - grad\ T^* - grad\ c^* = 0 \quad (12)$$
$$-\eta_t^{10}[\chi_3\, div\ \boldsymbol{u}^* + \chi_4 C^*] + (\nabla^2 - \eta_t^{10})T^* = 0 \quad (13)$$
$$\chi_5 \nabla^2\, div\ \boldsymbol{u}^* + \chi_6 \nabla^2 T^* + [\eta_c^{10} - (\chi_7 + \varepsilon_1^2 \eta_c^{10})\nabla^2]C^* = 0 \quad (14)$$

where

$$\eta_t^{10} = -i\omega(1 - i\omega\eta_0),\ \ \eta_c^{10} = -i\omega(1 - i\omega\eta)$$

Define matrix differential operator

$$\boldsymbol{F}(\boldsymbol{D_X}) = [F_{mn}(\boldsymbol{D_X})]_{4\times4} \quad (15)$$

where

$$F_{m1}(\boldsymbol{D_X}) = [(\chi_1 - \omega^2 \varepsilon_1^2)\nabla^2 + \omega^2]\delta_{m1} + \chi_2 \frac{\partial^2}{\partial x \partial y^*}$$
$$F_{m2}(\boldsymbol{D_X}) = [(\chi_1 - \omega^2 \varepsilon_1^2)\nabla^2 + \omega^2]\delta_{m2} + \chi_2 \frac{\partial^2}{\partial z \partial y^*}$$
$$F_{m3}(\boldsymbol{D_X}) = F_{m4}(\boldsymbol{D_X}) = -\frac{\partial}{\partial y^*},\ \ F_{3n}(\boldsymbol{D_X}) = -\eta_t^{10} \chi_3 \frac{\partial}{\partial y^*}$$
$$F_{4n}(\boldsymbol{D_X}) = \chi_5 \nabla^2 \frac{\partial}{\partial y^*},\ \ F_{33}(\boldsymbol{D_X}) = \nabla^2 - \eta_t^{10},\ \ F_{34}(\boldsymbol{D_X}) = \eta_t^{10} \chi_4$$
$$F_{43}(\boldsymbol{D_X}) = \chi_2 \nabla^2,\ \ F_{44}(\boldsymbol{D_X}) = -\chi_1 \nabla^2 + (1 - \varepsilon_1^2 \nabla^2)\eta_c^{10};\quad m, n = 1,2$$

For $m = n = 1$, $y^* = x$ and for $m = n = 2$, $y^* = z$; δ_{mn} is the Kronecker’s delta.

The system of equations 12 − 14 can be written as

$$\boldsymbol{F}(\boldsymbol{D_X})\boldsymbol{B}(\boldsymbol{X}) = \boldsymbol{0} \quad (16)$$

where $\boldsymbol{B} = (u_1^*, u_3^*, T^*, C^*)$ is a four component vector function.

Assume that

$$-(\chi_1 + \chi_2 - \omega^2 \varepsilon_1^2)(\chi_1 - \omega^2 \varepsilon_1^2)(\chi_1 + \varepsilon_1^2 \eta_c^{10}) \neq 0 \quad (17)$$

If condition (17) is satisfied then $\boldsymbol{F}$ is an elliptic differential operator (Hörmander [15]).

Definition. Fundamental solution of system of equations $12-14$ is the matrix $\boldsymbol{V}(\boldsymbol{X})=[V_{ij}(\boldsymbol{X})]_{4\times4}$ satisfying the condition

$$\boldsymbol{F}(\boldsymbol{D_X})\boldsymbol{V}(\boldsymbol{X})=\boldsymbol{\delta}(\boldsymbol{X})\boldsymbol{I}(\boldsymbol{X}) \tag{18}$$

where $\boldsymbol{\delta}$ is the Dirac delta and $\boldsymbol{I}=[\boldsymbol{\delta}_{ij}]_{4\times4}$ is the unit matrix.
We now construct $\boldsymbol{V}(\boldsymbol{X})$ in terms of elementary functions.

3. FUNDAMENTAL SOLUTION OF SYSTEM OF EQUATIONS OF STEADY OSCILLATIONS

We consider the following system of equations

$$[(\chi_1-\omega^2\varepsilon_1^2)\nabla^2+\omega^2]\boldsymbol{u}^*+\chi_2\, grad\, div\, \boldsymbol{u}^*-\eta_t^{10}\chi_3\, grad\, T^*+\chi_5\nabla^2 grad\, c^*=\boldsymbol{L}' \tag{19}$$

$$-div\,\boldsymbol{u}^*+(\nabla^2-\eta_t^{10})T^*+\chi_2\nabla^2C^*=M \tag{20}$$

$$-\,div\,\boldsymbol{u}^*-\eta_t^{10}\chi_4T^*+[-\chi_7\nabla^2+(1-\varepsilon_1^2\nabla^2)\eta_c^{10}]C^*=N \tag{21}$$

where $\boldsymbol{L}'$ is a vectors function on E^3 and M,N denote scalar functions.
The system of equations $19-21$ may be written in the form

$$\boldsymbol{F}^t(\boldsymbol{D_X})\,\boldsymbol{B}(\boldsymbol{X})=\boldsymbol{G}(\boldsymbol{X}) \tag{22}$$

where $\boldsymbol{F}^t$ is the transpose of F and $\boldsymbol{G}=(\boldsymbol{L}',M,N)$**.** Applying operator div to the equations (19), we get

$$[(1-\omega^2\varepsilon_1^2)\nabla^2+\omega^2]\,div\,\boldsymbol{u}^*-\eta_t^{10}\chi_3\nabla^2\,T^*+\chi_5\nabla^4c^*=div\,\boldsymbol{L}' \tag{23}$$

$$-div\,\boldsymbol{u}^*+(\nabla^2-\eta_t^{10})T^*+\chi_2\nabla^2C^*=M \tag{24}$$

$$-\,div\,\boldsymbol{u}^*-\eta_t^{10}\chi_4T^*+[-\chi_7\nabla^2+(1-\varepsilon_1^2\nabla^2)\eta_c^{10}]C^*=N \tag{25}$$

Equations $23-25$ may be expressed as

$$\boldsymbol{J}(\nabla^2)\boldsymbol{E}=\boldsymbol{Q} \tag{26}$$

where

$$\boldsymbol{E}=(div\,\boldsymbol{u}^*,T^*,C^*),\quad \boldsymbol{Q}=(div\,\boldsymbol{L}',M,N)=(j_1,j_2,j_3)$$

and

$$\boldsymbol{J}(\nabla^2)=[J_{mn}(\nabla^2)]_{3\times3}$$
$$=\begin{bmatrix}(1-\omega^2\varepsilon_1^2)\nabla^2+\omega^2 & -\eta_t^{10}\chi_3\nabla^2 & \chi_5\nabla^4\\ -1 & \nabla^2-\eta_t^{10} & \chi_2\nabla^2\\ -1 & -\eta_t^{10}\chi_4 & -\chi_7\nabla^2+(1-\varepsilon_1^2\nabla^2)\eta_c^{10}\end{bmatrix}_{3\times3} \tag{27}$$

Equations $23-25$ may be expressed as

$$\varphi_1(\nabla^2)\boldsymbol{E}=\boldsymbol{\Omega} \tag{28}$$

where

$$\boldsymbol{\Omega}=(\Omega_1,\Omega_2,\Omega_3),\quad \Omega_n=e_1^*\sum_{m=1}^{3}J_{mn}^*j_m$$

$$\varphi_1(\nabla^2)=e_1^*\det\boldsymbol{J}(\nabla^2),\quad e_1^*=-\frac{1}{(1-\omega^2\varepsilon_1^2)(\chi_7+\varepsilon_1^2\eta_c^{10})} \tag{29}$$

and J_{mn}^* is the cofactor of element J_{mn} of matrix $\boldsymbol{J}$**.**
From equations 27 and 29, we have

$$\varphi_1(\nabla^2)=\prod_{m=1}^{3}(\nabla^2+\mu_m^2)$$

where $\mu_m^2,\ m=1,2,3$ are roots of equation $\varphi_1(-r)=0$ (with respect to r).
Applying the operator $\varphi_1(\nabla^2)$ to the equation 12 and using equation 28, we obtain

$$\varphi_1(\nabla^2)[(\chi_1-\omega^2\varepsilon_1^2)\nabla^2+\omega^2]\boldsymbol{u}^*=-\chi_2\,grad\,\Omega_1+grad\,\Omega_2+grad\,\Omega_3 \tag{30}$$

Above equation can also be written as

$$\varphi_1(\nabla^2)\ \varphi_2(\nabla^2)\boldsymbol{u}^*=\boldsymbol{\Omega}' \tag{31}$$

where

$$\varphi_2(\nabla^2)=(\chi_1-\omega^2\varepsilon_1^2)\nabla^2+\omega^2 \tag{32}$$

and

$$\boldsymbol{\Omega}'=-\chi_2\,grad\,\Omega_1+grad\,\Omega_2+grad\,\Omega_3 \tag{33}$$

It may be written as

$$\varphi_2(\nabla^2)=(\nabla^2+\mu_4^2) \tag{34}$$

where μ_4^2 is a root of equation $\varphi_2(-r)=0$ (with respect to r).
From equations 28 and 31, we have

$$\boldsymbol{\sigma}(\nabla^2)\boldsymbol{B}(\boldsymbol{X})=\widehat{\boldsymbol{\Omega}}(\boldsymbol{X}) \tag{35}$$

where

$$\widehat{\boldsymbol{\Omega}}=(\boldsymbol{\Omega}',\Omega_2,\Omega_3\,)$$

and

$$\boldsymbol{\sigma}(\nabla^2)=[\sigma_{gh}(\nabla^2)]_{4\times4}$$
$$\sigma_{uu}(\nabla^2)=\ \varphi_1(\nabla^2)\ \varphi_2(\nabla^2)$$
$$\sigma_{33}(\nabla^2)=\sigma_{44}(\nabla^2)=\ \varphi_1(\nabla^2),\quad \sigma_{gh}(\nabla^2)=0$$
$$u=1,2;\quad g,h=1,2,3,4\,;\quad g\neq h;$$

Equations 26, 29 and 33 can also be written as

$$\boldsymbol{\Omega}'=c_{11}(\nabla^2)\,grad\,div\,\boldsymbol{L}'+c_{21}(\nabla^2)\,grad\,M+c_{31}(\nabla^2)\,grad\,N \tag{36}$$

$$\Omega_2 = c_{12}(\nabla^2)\, div\, \boldsymbol{L}' + c_{22}(\nabla^2)M + c_{32}(\nabla^2)N \tag{37}$$
$$\Omega_3 = c_{13}(\nabla^2)\, div\, \boldsymbol{L}' + c_{23}(\nabla^2)M + c_{33}(\nabla^2)N \tag{38}$$

where

$$c_{11}(\nabla^2) = -\chi_2 e_1^* J_{11}^* + e_1^* J_{12}^* + e_1^* J_{13}^*,\ \ c_{12}(\nabla^2) = e_1^* J_{12}^*,\ \ c_{13}(\nabla^2) = e_1^* J_{13}^*$$
$$c_{21}(\nabla^2) = -\chi_2 e_1^* J_{21}^* + e_1^* J_{22}^* + e_1^* J_{23}^*,\ \ c_{22}(\nabla^2) = e_1^* J_{22}^*,\ \ c_{23}(\nabla^2) = e_1^* J_{23}^*$$
$$c_{31}(\nabla^2) = -\chi_2 e_1^* J_{31}^* + e_1^* J_{32}^* + e_1^* J_{33}^*,\ \ c_{32}(\nabla^2) = e_1^* J_{32}^*,\ \ c_{33}(\nabla^2) = e_1^* J_{33}^*$$

From equations $36 - 38$, we get

$$\widehat{\boldsymbol{\Omega}}(\boldsymbol{X}) = \boldsymbol{R}^t(\boldsymbol{D_X})\boldsymbol{G}(\boldsymbol{X}) \tag{39}$$

where

$$\boldsymbol{R} = [R_{gh}]_{4\times 4}$$
$$R_{m1}(\boldsymbol{D_X}) = c_{11}(\nabla^2)\frac{\partial^2}{\partial x \partial y^*},\ \ R_{m2}(\boldsymbol{D_X}) = c_{11}(\nabla^2)\frac{\partial^2}{\partial z \partial y^*},\ \ R_{m3}(\boldsymbol{D_X}) = c_{21}(\nabla^2)\frac{\partial}{\partial y^*}$$
$$R_{m4}(\boldsymbol{D_X}) = c_{31}(\nabla^2)\frac{\partial}{\partial y^*},\ \ R_{3n}(\boldsymbol{D_X}) = c_{12}(\nabla^2)\frac{\partial}{\partial y^*},\ \ R_{4n}(\boldsymbol{D_X}) = c_{13}(\nabla^2)\frac{\partial}{\partial y^*}$$
$$R_{33}(\boldsymbol{D_X}) = c_{22}(\nabla^2),\ \ R_{34}(\boldsymbol{D_X}) = c_{32}(\nabla^2),\ \ R_{43}(\boldsymbol{D_X}) = c_{43}(\nabla^2)$$
$$R_{44}(\boldsymbol{D_X}) = c_{44}(\nabla^2); \qquad m, n = 1,2 \tag{40}$$

For $m = n = 1,\ y^* = x$ and for $m = n = 2,\ y^* = z$

From equations 22, 31 and 35, we obtain

$$\boldsymbol{\sigma B} = \boldsymbol{R}^t \boldsymbol{F}^t \boldsymbol{B}$$

Above equation may be rewritten as

$$\boldsymbol{R}^t \boldsymbol{F}^t = \boldsymbol{\sigma}$$

Therefore, we get

$$\boldsymbol{F}(\boldsymbol{D_X})\boldsymbol{R}(\boldsymbol{D_X}) = \boldsymbol{\sigma}(\nabla^2) \tag{41}$$

Assume that

$$\mu_p^2 \neq \mu_q^2 \neq 0, \quad p, q = 1,2,3,4; \quad p \neq q$$

We now define the matrix

$$\boldsymbol{H}(\boldsymbol{X}) = [H_{rs}(\boldsymbol{X})]_{4\times 4}$$
$$H_{pp}(\boldsymbol{X}) = \sum_{n=1}^{4} r_{1n}\tau_n(\boldsymbol{X}), \quad H_{33}(\boldsymbol{X}) = H_{44}(\boldsymbol{X}) = \sum_{n=1}^{3} r_{2n}\tau_n(\boldsymbol{X}),\ \ H_{uv}(\boldsymbol{X}) = 0$$
$$p = 1,2, \quad u, v = 1,2,3,4; \quad u \neq v$$

where

$$\tau_n(\boldsymbol{X}) = -\frac{1}{4\pi|\boldsymbol{X}|}\exp(i\mu_n|\boldsymbol{X}|), \quad n = 1,2,3,4$$
$$r_{1u} = \prod_{m=1, m\neq u}^{4}(\mu_m^2 - \mu_u^2)^{-1}, \quad u = 1,2,3,4$$
$$r_{2l} = \prod_{m=1, m\neq l}^{3}(\mu_m^2 - \mu_l^2)^{-1}, \quad l = 1,2,3 \tag{42}$$

Now, we prove the following Lemma

Lemma: The matrix $\boldsymbol{H}$ defined above is the fundamental matrix of operator $\boldsymbol{\sigma}(\nabla^2)$, that is

$$\boldsymbol{\sigma}(\nabla^2)\boldsymbol{H}(\boldsymbol{X}) = \delta(\boldsymbol{X})\boldsymbol{I}(\boldsymbol{X}) \tag{43}$$

Proof: To prove the Lemma, it is sufficient to show that

$$\varphi_1(\nabla^2)\,\varphi_2(\nabla^2)H_{11}(\boldsymbol{X}) = \delta(\boldsymbol{X})$$
$$\varphi_1(\nabla^2)H_{33}(\boldsymbol{X}) = \delta(\boldsymbol{X}) \tag{44}$$

Consider

$$r_{21} + r_{22} + r_{23} = \frac{-s_1 + s_2 - s_3}{s_4}$$

where

$$s_1 = (\mu_2^2 - \mu_3^2),\ \ s_2 = (\mu_1^2 - \mu_3^2),\ \ s_3 = (\mu_1^2 - \mu_2^2)$$
$$s_4 = (\mu_1^2 - \mu_2^2)(\mu_1^2 - \mu_3^2)(\mu_2^2 - \mu_3^2)$$

Solving above relations, we have

$$r_{21} + r_{22} + r_{23} = 0 \tag{45}$$

Similarly from equation 42 we can also find out

$$r_{22}(\mu_1^2 - \mu_2^2) + r_{23}(\mu_1^2 - \mu_3^2) = 0 \tag{46}$$
$$r_{23}(\mu_1^2 - \mu_3^2)(\mu_2^2 - \mu_3^2) = 1 \tag{47}$$

Also, we can write

$$(\nabla^2 + \mu_m^2)\tau_n(\boldsymbol{X}) = \delta(\boldsymbol{X}) + (\mu_m^2 - \mu_n^2)\tau_n(\boldsymbol{X}), \qquad m, n = 1,2,3 \tag{48}$$

Consider

$$\varphi_1(\nabla^2)H_{33}(\boldsymbol{X}) = (\nabla^2+\mu_1^2)(\nabla^2+\mu_2^2)(\nabla^2+\mu_3^2)\sum_{n=1}^{3} r_{2n}\tau_n(\boldsymbol{X})$$
$$= (\nabla^2+\mu_2^2)(\nabla^2+\mu_3^2)\sum_{n=1}^{3} r_{2n}[\delta(\boldsymbol{X}) + (\mu_1^2-\mu_n^2)\tau_n(\boldsymbol{X})]$$
$$= (\nabla^2+\mu_2^2)(\nabla^2+\mu_3^2)\left[\delta(\boldsymbol{X})\sum_{n=1}^{3} r_{2n} + \sum_{n=2}^{3} r_{2n}\,(\mu_1^2-\mu_n^2)\tau_n(\boldsymbol{X})\right]$$
$$= (\nabla^2+\mu_2^2)(\nabla^2+\mu_3^2)\sum_{n=2}^{3} r_{2n}\,(\mu_1^2-\mu_n^2)\tau_n(\boldsymbol{X})$$
$$= (\nabla^2+\mu_3^2)\sum_{n=2}^{3} r_{2n}\,(\mu_1^2-\mu_n^2)\,[\delta(\boldsymbol{X}) + (\mu_2^2-\mu_n^2)\tau_n(\boldsymbol{X})]$$
$$= (\nabla^2+\mu_3^2)\sum_{n=3}^{3} r_{2n}\,(\mu_1^2-\mu_n^2)(\mu_2^2-\mu_n^2)\tau_n(\boldsymbol{X})$$
$$= (\nabla^2+\mu_3^2)\,r_{23}\,(\mu_1^2-\mu_3^2)(\mu_2^2-\mu_3^2)\tau_3(\boldsymbol{X})$$
$$= (\nabla^2+\mu_3^2)\tau_3(\boldsymbol{X})$$
$$= \delta(\boldsymbol{X})$$

Similarly, equation $(44)_1$ can be proved.
Now, Define the matrix

$$\boldsymbol{V}(\boldsymbol{X}) = \boldsymbol{R}(\boldsymbol{D}_{\boldsymbol{X}})\boldsymbol{H}(\boldsymbol{X}) \tag{49}$$

Using equations 41, 43 and 49, we obtain

$$\boldsymbol{F}(\boldsymbol{D}_{\boldsymbol{X}})\,\boldsymbol{V}(\boldsymbol{X}) = \boldsymbol{F}(\boldsymbol{D}_{\boldsymbol{X}})\boldsymbol{R}(\boldsymbol{D}_{\boldsymbol{X}})\boldsymbol{H}(\boldsymbol{X}) = \boldsymbol{\sigma}(\nabla^2)\boldsymbol{H}(\boldsymbol{X}) = \delta(\boldsymbol{X})\boldsymbol{I}(\boldsymbol{X}) \tag{50}$$

Therefore, $\boldsymbol{V}(\boldsymbol{X})$ is solution to equation 18.
Hence, we proved the following Theorem
Theorem. The matrix $\boldsymbol{V}(\boldsymbol{X})$ defined by the equation 49 is the fundamental solution of system of equations $12-14$.

4. BASIC PROPERTIES OF THE MATRIX $V(X)$

Property 1. Every column of the matrix $\boldsymbol{V}(\boldsymbol{X})$ is the solution of the equations 12−14 for all points $\boldsymbol{X}\in E^3$ except the origin.
Property 2. The matrix $\boldsymbol{V}(\boldsymbol{X})$ can be written as

$$\boldsymbol{V} = [V_{rs}]_{4\times4}$$
$$V_{pq}(\boldsymbol{X}) = R_{pq}(\boldsymbol{D}_{\boldsymbol{X}})H_{11}(\boldsymbol{X})$$
$$V_{pn}(\boldsymbol{X}) = R_{pn}(\boldsymbol{D}_{\boldsymbol{X}})H_{33}(\boldsymbol{X})$$
$$p = 1,2,3,4;\quad q = 1,2;\quad n = 3,4.$$

5. SPECIAL CASE

If we neglect the nonlocal parameter $(\varepsilon' = 0)$ in the equations 12−14, we obtain the system of equations of steady state oscillations for homogenous isotropic generalized thermoelastic solid with diffusion as

$$[\chi_1\nabla^2+\omega^2]\boldsymbol{u}^* + \chi_2\, grad\; div\; \boldsymbol{u}^* - grad\; T^* - grad\; c^* = 0 \tag{51}$$
$$-\eta_t^{10}[\chi_3\; div\; \boldsymbol{u}^* + \chi_4 C^*] + (\nabla^2 - \eta_t^{10})T^* = 0 \tag{52}$$
$$\chi_5\nabla^2\; div\; \boldsymbol{u}^* + \chi_6\nabla^2 T^* + [\eta_c^{10} - \chi_7\nabla^2]C^* = 0 \tag{53}$$

The fundamental solution of above system of equations is similar as obtained by Kumar and Kansal [24].

6. CONCLUSION

The fundamental solution of system of equations for steady oscillations has been constructed in the case of generalized theory of nonlocal homogenous isotropic thermoelastic diffusion. The analysis of fundamental solution $\boldsymbol{V}(\boldsymbol{X})$ of the system of equations 12-14 is beneficial to investigate various three dimensional problems of nonlocal homogenous isotropic elastic solid with diffusion. Also this study is very useful in different fields of geophysics and electronics like seismology, telecommunication etc.

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