# Hand Shaking Lemma Type Results For Blocks

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**Abstract:The origine of graph theory theory dates back to 1957 with the invention of solution to Konig's berg bridge problem by father of graph theory Euler. Then the first theorem in graph theory which some times called as Hand shaking lemma by virtue of its property states that sum of all the degrees is equal to twice the number of edges.**

**Index Terms: Component, formatting, style, styling, insert.** *(key words)*

## I. INTRODUCTION

For any undefined terminologies we refer [1] and [2]. We denote any graph *G(V,E)* with |*V*|=*p* and |*E*|=*q*. Degree of a vertex *v*, denoted as *d*(*v*) is the number of edges incident on *v*. Similarly, the degree of an edge $x = uv$, denoted as $d_e(x)$ is the number of edges adjacent to the edge *x*. Equivalently, $d_e(x) = d(u) + d(v) - 2$. The maximum degree, the minimum degree, the maximum edge degree and the minimum edge degree of *G* are respectively denoted by $\Delta(G)$, $\delta(G)$, $\Delta_e(G)$, $\delta_e(G)$. If $\Delta(G) = \delta(G) = k$, then *G* is said to be *k-regular*. The origine of graph theory theory dates back to 1957 with the invention of solution to Konig's berg bridge problem by father of graph theory Euler. Then the first theorem in graph theory which some times called as Hand shaking lemma by virtue of its property states that

**Theorem 1(Hand shaking lemma or Forst theorem in graph theory**) [1]: *For any graph G(V,E) with p vertices and q edges*

$$\sum_{v \in V} d(v) = 2q$$

A vertex $v \in V$ is a *cutvertex* of a graph *G* if *G* - *v* is disconnected and such an edge is a *bridge or a cutedge*. A graph *G* is *separable* if it has a cutvertex otherwise it is *non-separable*. A maximal non-separable subgraph is a *block* of *G*. A maximal complete subgraph is a *clique*. Let *B*(*G*) and *C*(*G*) denote the set of all blocks and cutvertices of *G* respectively. All through the discussion *p* and *q* be the order and size of the graph while *m* and *n* are spared for number of blocks and number of cutvertices in *G*. If a block $b \in B(G)$ contains a cutvertex $c \in C(G)$ then we say that *b* and *c* incident to each other. Two blocks in *G* are adjacent if there is a common cutvertex incident with them. On the other hand two cutvertices are adjacent if there is a common block incident with them. In this paper we define few new degree concepts on blocks and try to establish Handshaking lemma type results in the sense that we tried to find an expression for some block degrees. A detailed study is available in [3] [4] [5] [6] and [7].

## II. BLOCK-DEGREE AND CUTVERTEX DEGREE

Here we introduce some new block-degree concepts.

**Definition 2.3.1** Two vertices $u,w \in V$ are vv-adjacent if they incident on the same block. Then vv-degree of $u = d_{vv}(u)$ is the number of vertices vv-adjacent to *u*. Then $\Delta_{vv}(G)$ and $\delta_{vv}(G)$ denote the maximum and minimum *vv*-degrees of *G*.

**Definition 2.3.2** *vb-degree* (vertex block-degree) $d_{vb}(u)$ of a vertex *u*, is the number of blocks incident on *u*. For any noncutvertex $v \in V$, $d_{vb}(v) = 1$ and for any cutvertex $c \in C(G)$, $d_{vb}(c) \geq 2$.

*bv-degree* (*block vertex-degree*) *of a block f*, $d_{bv}(f)$ is the number of vertices in the block *f* and *be-degree* (*block edge- degree*) *of a block f*, $d_{be}(f)$ is the number of edges in the block *f*. *bb-degree* (*block block-degree) of a block h*, $d_{bb}(h)$ is the number of blocks adjacent to *h*. Finally, *cutvertex-degree of a block b*, $d_c(b)$ is the number of cutvertices incident on *b*. If *b* is a *pendant block* then $d_c(b) = 1$. For any non-pendant block $d_c(b) \geq 2$. Then $\Delta_{vb}(G)$, $\delta_{vb}(G)$, $\Delta_{bv}(G)$, $\delta_{bv}(G)$, $\Delta_{be}(G)$, $\delta_{be}(G)$, $\Delta_{bb}(G)$, $\delta_{bb}(G)$ $\Delta_c(G)$ and $\delta_c(G)$ are similarly defined.

## III. SUM OF BLOCK DEGREES

The following results appear in [69 and 70]. Those results in the block degree terminologies read as follows.

**Proposition 3.1** *For any graph G, with m blocks and n cutvertices,*

$$\sum_{u \in V}(d_{vb}(u) - 1) = m - 1 \qquad (1)$$

$$\sum_{h \in B(G)}(d_c(h) - 1) = n - 1 \qquad (2)$$

**Corollary 3.1.1** *For any graph G with p vertices* $\sum_{h \in B(G)}(d_{bv}(h) - 1) = p - 1$

**Proof.** Infact, $d_{bv}(h) = d_c(h)$ + number of non cutvertices of *h*. Noting that there are *p-n* non cutvertices in a graph we have, $\sum_{h \in B(G)}(d_{bv}(h) - 1) = \sum_{h \in B(G)}(d_c(h) - 1) + p - n = n - 1 + p - n = p - 1$ using equation (2). □

It is observed that sum of all vb-degree of all cutvertices and sum of all cutvertex degree of all blocks are equal.

**Theorem 3.2** *For any graph G, with m blocks and n cutvertices,*

$$\sum_{c \in C(G)} d_{vb}(c) = \sum_{h \in B(G)} d_c(h) = m + n - 1 \qquad (3)$$

**Proof.** Since for each non cutvertex $d_{vb}(u) = 1$, the noncutvertices contribute null to the sum in (1) and hence can be written as $m - 1 = \sum_{c\in C(G)}(d_{vb}(c) - 1)$. Therefore $\sum_{c\in C(G)} d_{vb}(c) = m + n - 1$. Now from the Gallai's result (2), we have $n - 1 = \sum_{h\in B(G)}(d_c(h)) - m$. This yields the desired result (3). □

It is also true that sum of bv-degree of all blocks and sum of vb-degree of all vertices are equal which is proved in the next theorem.

**Theorem 3.3** *For any graph G, with m blocks and n cutvertices,*

$$\sum_{h\in B(G)} d_{bv}(h) = \sum_{u\in V} d_{vb}(u) = p + m - 1 \qquad (4)$$

**Proof.** From (1), $\sum_{u\in V}(d_{vb}(u) - 1) = \sum_{u\in V} d_{vb}(u) - p = m - 1$. Hence $\sum_{u\in V} d_{vb}(u) = p + m - 1$. Also from Corollary 3.1.1, $\sum_{h\in B(G)}(d_{bv}(h) - 1) = \sum_{h\in B(G)} d_{bv}(h) - m = p - 1$. Hence $\sum_{h\in B(G)} d_{bv}(h) = p + m - 1$. □

**Remark** It is immediate that $\sum_{h\in B(G)} d_{be}(h) = q$.

It is well known that the sum of edge degrees of all edges is given by $\sum_{x\in X(G)} d_e(x) = \sum_{u\in V(G)}(d(u))^2 - 2q$ (see Harary [1] ). Analogously, an expression for sum of bb-degree of all blocks is obtained in the next proposition.

**Proposition 3.4.** *Let G be any graph and $q_b$ denote the number of edges in the block graph $B_G(G)$, Then,*

$$\sum_{h\in B(G)} d_{bb}(h) = 2q_b = \left[\sum_{c\in C(G)}[(d_{vb}(c))^2 - (m + n - 1)\right] \qquad (5)$$

**Proof.** Since the vertices of $B_G(G)$, are blocks of $G$, we have

$\sum_{h\in B(G)} d_{bb}(h) = \sum_{u\in V(B_G(G))} d(u) = 2q_b$. As every block in $B_G(G)$ is a clique, each cutvertex $c\epsilon G$ yield $\binom{d_{vb}(c)}{2}$ edges in $B_G(G)$. Hence

$q_b = \sum_{c\in C(G)}\binom{d_{vb}(c)}{2} = \frac{1}{2}\sum_{c\in C(G)}[(d_{vb}(c))^2 - d_{vb}(c)] = \frac{1}{2}\left[\sum_{c\in C(G)}(d_{vb}(c))^2 - (m + n - 1)\right]$.

Then the result follows. □

In the next result a lower bound for sum of the squares of vb-degree of all cutvertices is obatined

**Proposition 3.5.** *For any graph G, with m blocks and n cutvertices,*

$$\frac{(m+n-1)^2}{n} \le \sum_{c\in C(G)}(d_{vb}(c))^2 \qquad (6)$$

**Proof.** From Cauchy - Schwarz inequality we have $\sum_{i=1}^{n}|a_i b_i| \le \sqrt{\sum_{i=1}^{n}|a_i^2|}\sqrt{\sum_{i=1}^{n}|b_i^2|}$ where $a_i$ and $b_i$ are integers.

Taking $a_i = d_{vb}(c_i)$ and $b_i = 1$ in the above inequality we get

$(\sum_{i=1}^{n} d_{vb}(c_i))^2 \le n\sum_{i=1}^{n}(d_{vb}(c_i))^2$. Then the result (6) follows from equation (3). □

In the next proposition we obtain a lower bound for sum of bb degree of all blocks in terms of number of blocks and number of cutvertices.

**Proposition 3.6** *For any graph G, with m blocks and n cutvertices,*

$$\frac{(m-1)(m+n-1)}{n} \le \sum_{h\in B(G)} d_{bb}(h) \qquad (7)$$

Further, this bound is sharp.

**Proof.** From Equations (2.5) and (2.6) we have ,

$$\sum_{h\in B(G)} d_{bb}(h) = \sum_{c\in C(G)}(d_{vb}(c))^2 - (m + n - 1) \ge \frac{(m+n-1)^2}{n} - (m + n - 1)$$

$$= \frac{m^2+n^2+1+2mn-2n-2m-mn-n^2+n}{n}$$

$$= \frac{(m-1)^2+n(m-1)}{n} = \frac{(m-1)(m+n-1)}{n}.$$

The bound is sharp is evident from the fact that any B-path and B-complete graphs attain the bound. □

## IV POINT GRAPH AND SUM OF VV-DEGREE

Similar to cutvertex graph we define *point graph* $P_G(G)$ of a graph whose vertex set is same as that of $G$ and any two vertices in $P_G(G)$ are adjacent if and only if they are vv-adjacent in $G$. We observe that $P_G(P_G(G)) = P_G(G)$. One would ask when can we say that the given graph is the point graph of some graph? The next question that may arise is under what conditions $P_G(G)$ is isomorphic to $G$ ? A point graph also admits same characterization as that of cutvertex graph. The answers for both the questions are straight forward and therefore we state the results without proof in the next proposition.

**Proposition 4.1**

(i) *A graph G is a point graph of some graph if and only if every block of G is complete.*

(ii) *$P_G(G) \cong G$ if and only if every block of G is complete.*

**Remark.** For any tree $T$, $P_G(T) \cong T$

It is quite natural to ask the question what is the number of edges in $P_G(G)$? It is answered in the next result.

**Proposition 4.2** *Let G be any graph and $q_p$ denote the number of edges in the point graph $P_G(G)$. Then*

$$\sum_{u\in V} d_{vv}(u) = 2q_p = \sum_{h\in B(G)}(d_{bv}(h))^2 - (p + m - 1) \qquad (8)$$

**Proof.** Since vv-degree of a vertex is the degree of the corresponding vertex in $P_G$ $(G)$, we have $\sum_{u\in V(G)} d_{vv}(u) = \sum_{u\in V(P_G(G))} d(u) = 2q_p$.

As every block in $P_G$ $(G)$ is a clique, every block in $G$ yield, $\binom{d_{bv}(h)}{2}$ edges in $P_G$ $(G)$. Then, $q_b = \sum_{h\in B(G)} \binom{d_{bv}(h)}{2} = \frac{1}{2}\sum_{h\in B(G)}[(d_{bv}(h))^2 - d_{bv}(h)] = \frac{1}{2}\left[\sum_{h\in B(G)}(d_{bv}(h))^2 - (p+m-1)\right]$ using equation (2.4). Then the result (8) follows. □

**Proposition 4.3** *For any (p, q) graph G, with m blocks and n cutvertices,*

$$\frac{(p+m-1)^2}{m} \le \sum_{h\in B(G)}(d_{bv}(h))^2 \qquad (9)$$

**Proof.** Taking $a_i = d_{bv}(h)$ and $b_i = 1$ in the Cauchy - Schwarz inequality used in Proposition 4.2, we get $(\sum_{i=1}^{m} d_{bv}(h))^2 \le m\sum_{i=1}^{m}(d_{bv}(h))^2$. Then the result follows from equation (4).

**Proposition 4.3** *For any (p, q) graph G, with m blocks and n cutvertices,*

$$\frac{(p-1)(p+m-1)}{m} \le \sum_{u\in V} d_{vv}(u) \qquad (10)$$

Further, this bound is sharp.

**Proof.** From equation (8) and (9) we have,

$$\sum_{h\in B(G)} d_{vv}(u) = \sum_{h\in B(G)}(d_{bv}(h))^2 - (p+m-1) \ge \frac{(p+m-1)^2}{m} - (p+m-1)$$
$$= \frac{m^2+p^2+1+2mp-2p-2m-mp-m^2+m}{m}$$
$$= \frac{(p-1)^2+m(p-1)}{m} = \frac{(p-1)(p+m-1)}{m}.$$

The bound is attained for any B-complete graph. □

## V BOUNDS ON BLOCKS AND CUTVERTICES

The following result gives an estimate for number of blocks in any graph. One can refer for further results [8] [9] [10] [11] and [12].

**Proposition 5.1** *For any graph G, with m blocks and n cutvertices,*

$$\frac{q}{\Delta_{be}} \le m \le \frac{q}{\delta_{be}} \qquad (11)$$

**Proof.** From Remark above, we have $\sum_{h\in B(G)} d_{be}(h) = q$.

Therefore $m\delta_{be} \le \sum_{h\in B(G)} d_{be}(h) = q \le m\Delta_{be}$. Hence $\frac{q}{\Delta_{be}} \le m \le \frac{q}{\delta_{be}}$ follows.

**Proposition 5.2** *For any graph G, with m blocks and n cutvertices,*

$$\frac{p-1}{\Delta_{bv}-1} \le m \le \frac{p-1}{\delta_{bv}-1} \qquad (12)$$

**Proof.** From equation (4) we have, $m\delta_{bv} \le \sum_{h\in B(G)} d_{bv}(h) = p+m-1 \le m\Delta_{bv}$. This equation on simplification yields (12).

For any tree $T$ with $p$ vertices $m = q = p - 1$, $\delta_{bv} = \Delta_{bv} = 2$ and $\delta_{be} = \Delta_{be} = 1$. Hence any tree attains both upper and lower bounds in (2.11) and (2.12). Thus the bounds are sharp. □

**Definition** Let $B_P(G)$ and $B_{NP}(G)$ denote the set of all pendant and non-pendant blocks of $G$ respectively. Let $m_P = |B_P(G)|$ and $m_{NP} = |B_{NP}(G)|$. If $g$ is any pendant block, then $d_c(g) = 1$. Therefore $\delta_c(G)$ is always equal to 1. Hence another parameter is introduced and is defined as $\delta_{NPc} = \min_{h\in B_{NP}(G)}\{d_c(\mathrm{h})\}$.

A bound for number of non-pendant blocks is obtained in the next result.

**Proposition 5.3** *For any graph G, with m blocks and n cutvertices,*

$$\frac{n-1}{\Delta_c-1} \le m_{NP} \le \frac{n-1}{\delta_{NPc}-1} \qquad (13)$$

**Proof.** It is known that $B_{NP}(G) \cup B_P(G) = B(G)$ and $B_{NP}(G) \cap B_P(G) = \phi$, hence $m = m_p + m_{NP}$. Further, for any pendant block $b$, $d_c(b) = 1$, therefore $\sum_{b\in B_p(G)} d_c(b) = m_p$. Thus $\sum_{b\in B_{NP}(G)} d_c(b) = \sum_{b\in B(G)} d_c(b) - \sum_{b\in B_p(G)} d_c(b) = m+n-1-m_p = m_{NP}+n-1$ (using equation (3)). Hence $m_{NP}\delta_{NPc} \le \sum_{b\in B_{NP}(G)} d_c(b) = m_{NP}+n-1 \le m_{NP}\Delta_c$ which yields (13). □

**Definition.** If $u$ is a non-cutvertex, then $d_{vb}(u) = 1$. Therefore $\delta_{vb}$ is always equal to 1. Hence another parameter is introduced which is defined as $\delta_{cvb} = \min_{c\in C(G)}\{d_{vb}(\mathrm{c})\}$.

We are now ready to find a bound for number of cutvertices in any graph.

**Proposition 5.4** *For any graph G, with m blocks and n cutvertices,*

$$\frac{m-1}{\Delta_{vb}-1} \le n \le \frac{m-1}{\delta_{cvb}-1} \qquad (2.14)$$

**Proof.** From equation (2.3), we have $n\delta_{cvb} \le \sum_{c\in C(G)} d_{vb}(c) = m+n-1 \le n\Delta_{Vb}$ which yields (2.14). □

## VI BLOCK REGULAR GRAPHS

The new degree concepts paved the way to define varieties of block regular graphs. A graph $G$ is said to be $BVk$- regular if $d_{bv}(h) = k$ for every $h \in B(G)$. A graph $G$ is $BEk$- regular if $d_{be}(h) = k$ for every $h\in B(G)$. Similarly, if $d_{vb}(c) = k$ for every cut-vertex $c\in C(G)$, then $G$ is a $VBk$- regular graph. If $d_{bb}(h) = k$, then $G$ is a $BBk$- regular graph. On the other hand, if $d_c(h) = k$ for every nonpendant block we say that $G$ is a $Ck$-regular graph. Finally, if $d_{vv}(c) = k$ for every cut-vertex $c\in C(G)$, then $G$ is a $VVk$- regular graph. Any path $P_n$ is a $BE1$, $BV2$, $C2$, $VB2$ and $VV$ 2-regular graph. Any B-path $BP_m$ in which every block has $k$ vertices and $r$ edges is a $BEr$, $BVk$, $C2$, $VB2$ and $VV(2k-2)$ regular graph. Any tree $T$ is a $BV2$ and BE1-regular graph.

The *cactus K(i)* for $i \in N$, has vertex set $V(K(i)) = \{a_k \mid k = 1, \ldots i+1\} \cup \{b_j \mid j = 1 \ldots 2i\}$ and the edge set $E(K(i)) = \{a_k b_{2k-1} \mid k = 1, \ldots i\} \cup \{a_k b_{2k}, a_{k+1} b_{2k} \mid k = 1, \ldots i\}$. *The Cactus K(4)* is in Fig.2.6.1 which is *BE4, BV4, C2* , *VB2* and *VV6* regular graph. We observe that in general any Cactus $K(i)$ is a B-path with $i$ blocks and all the blocks being the cycle $C_i$. Hence $K(i)$ is a *BEi*, *BVi*, *C2*, *VB2* and *VV*(2*i* - 2) – regular graph.

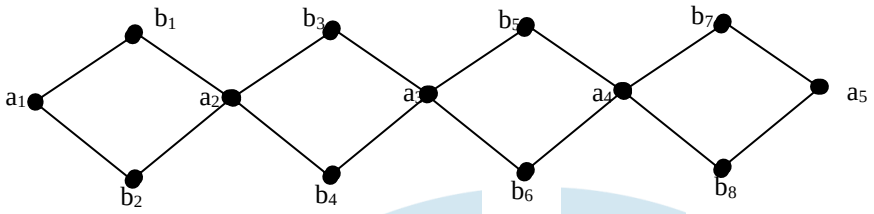


Fig 6.1
The Cactus K(4)

As another example, the graph $G$ in Fig.6.2, is a VB4, BV4,BE4,BB6, VV12 – regular graph on 67 vertices.

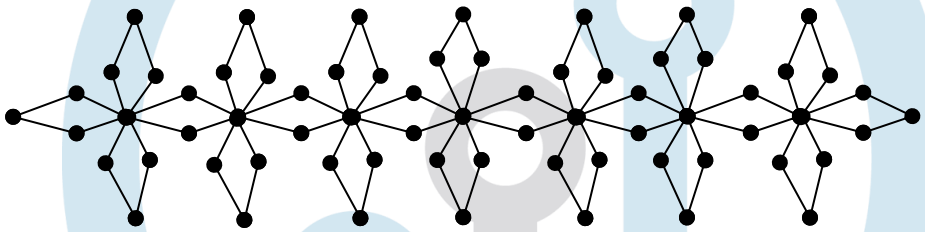

Fig. 6.2 VB4, BV4,BE4,V V12 – regular graph on 67 vertices.

**Proposition 6.1** *If G is a connected BVk- regular graph with $k \geq 3$ and m blocks, then*

*(i)* $p = m(k-1) + 1$

*(ii)* $mk \leq q \leq \frac{mk(k-1)}{2}$

**Proof.** If $G$ is a connected *BVk*- regular graph with $k \geq 3$, then every block of $G$ has exactly $k$ vertices. Then $mk = \sum_{h \in B(G)} d_{bv}(h) = p + m - 1$ from Theorem 2.4.3.
Therefore $p = m(k-1) + 1$ and the result (i) follows. A block with $k$ vertices with minimum number of edges is the cycle $C_k$, and every cycle $C_k$ has $k$ edges and hence $mk \leq q$. Similarly, a block with $k$ vertices with maximum number of edges is a complete graph $K_k$, which has $\binom{k}{2}$ edges, and thus we have $q \leq m\binom{k}{2}$ then the result follows. □

In the next proposition we characterize the *BVk*-regular graphs attaining $q = mk$ and $q = \frac{mk(k-1)}{2}$. The result being straight forward we omit the proof .

**Proposition 6.2** *If G is a connected BVk- regular graph with $k \geq 3$ and m blocks, then*

*(i)* $q = mk$ *if and only if every block of G is a cycle* $C_k$.

*(ii)* $q = \frac{mk(k-1)}{2}$ *if and only if every block of G is a clique* $Q_k$

**Proposition 6.3** *If G is a connected BEk- regular graph with m blocks, then*

*(i)* $q = mk$

*(ii)* $m(r-1)+1 \leq p \leq m(k-1)+1$ *where r is a least positive integer such that the* clique $Q_r$ *has k edges.*

**Proof.** Since $G$ is *BEk* -regular graph, every block of $G$ contains $k$ edges. A block containing $k$ edges with minimum number of vertices must be a clique $Q_r$ in $G$. If every block of $G$ contains atleast $r$ vertices then $mr \leq \sum_{h \in B(G)} d_{bv} = p + m - 1$ (Theorem 4.3) which yields lower bound in (ii). Similarly a block containing $k$ edges with maximum number of vertices is the cycle $C_k$ in $G$. If every block of $G$ contains atmost $k$ vertices then $mk \geq \sum_{h \in B(G)} d_{bv}(h) = p + m - 1$ (by Theorem 2.4.3) which yields upper bound in (ii). This completes the proof. □

**Proposition 6.4** Let *G be a BEk- regular graph with m blocks, then*

(i) *$p = m(r-1)+1$ if and only if every block of G is a clique $Q_r$ where r is a least positive integer such that the* clique *$Q_r$ has k edges.*

(ii) *$p = m(k-1) +1$ if and only if every block of G is a cycle $C_k$*

**Proof.** The result follows from the fact that a block containing *k* edges with minimum number of vertices must be a clique $Q_r$ and a block containing *k* edges with maximum number of vertices is a cycle $C_k$.

□

**Proposition 6.5** *A graph G is*

(i) *both BVk- regular and BEk- regular graph if and only if every block of G is $C_k$, a cycle with k vertices.*

(ii) *A graph G is BBk- regular if and only if G is B-complete*

**Proof.** The result (i) follows from the Proposition 2.6.4 and Proposition 2.6.6 The proof of (ii) is straight forward.

□

## VII APPLICATIONS:

B-regular graph structures are important in the study of biotechnology to get unique type of group cell structures. It finds its application in the study of regular structure of hydrocarbons in organic chemistry and crystal structure in crystallography. It can be used in design and architecture to get aesthetic view.


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