Relation and Function
(A Stepping Stone of Algebra)

Anshu Grade
Mathematics Teacher
Presentation Convent Sr. Sec. School
Delhi-110006

Abstract- When dealing with certain mathematical problems, it is sometimes necessary to have knowledge of “HOW TO DEAL WITH IT CORRECTLY”. For that we should know various powerful results to prove or disprove any mathematical statement. The main purpose of this paper is to provide several sufficient results ensuring that when a cartesian product is relation and when a relation is function, as well as some necessary conditions, together with several examples that show the applicability of these results.

Index Terms- domain, range, codomain, image.

INTRODUCTION

A relation is any association between elements of one set, called the domain or (less formally) the set of inputs, and another set, called the range or set of outputs. Some people mistakenly refer to the range as the codomain, but as we will see, that really means the set of all possible outputs—even values that the relation does not actually use. Formally, R is a relation if R ⊆ X × Y = {(x, y) | x ∈ X, y ∈ Y} for the domain X and codomain Y. In mathematics, a function is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output. Functions of various kinds are "the central objects of investigation" in most fields of modern mathematics. There are many ways to describe or represent a function. Some functions may be defined by a formula or algorithm that tells how to compute the output for a given input. Others are given by a picture, called the graph of the function. There are many operations on relations are inversion, concatenation, diagonal of set etc. There are three types of functions: injective functions, surjective functions, bijective functions. There are many applications of relations and functions. Relations can be transitive, symmetric and reflexive. In this research paper we will discuss how relations are different and related from the functions in discrete mathematics.

Relations:
A relation is used to describe certain properties of things. That way, certain things may be connected in some way; this is called a relation. It is clear, that things are either related, or they are not, there are no in between.

I. CARTESIAN PRODUCT

Consider two arbitrary sets X and Y. The set of all ordered pairs (x, y) where x ∈ X and y ∈ Y is called the product, or Cartesian product, of X and Y. A short designation of this product is X × Y, which is read “X cross Y.”

By definition,
X × Y= {(x, y) | x ∈ X and y ∈ Y}

Example Let X = {1, 2} and Y = {10, 15, 20}. Then,
X × Y = {(1,10), (1,15), (1,20), (2,10), (2,15), (2,20)}
Y × X = {(10,1), (15,1), (20,1), (10,2), (15,2), (20,2)}
Also, X× X = {(1,1), (1,2), (2,1), (2,2)}

One frequently writes X² instead of X × X.

There are two things worth noting in the above examples. First of all, X × Y ≠ Y × X. The Cartesian product deals with ordered pairs, so naturally the order in which the sets are considered is important.

Secondly, using n(S) for the number of elements in a set S, we have:
n (X× Y) = n(X)×n(Y) = 2×3 = 6

II. RELATION

A relation from a set X to a set Y is any subset of the Cartesian product X×Y
Definition Let $X$ and $Y$ be sets. Any set of ordered pairs $(x, y)$ called a relation in $x$ and $y$. Furthermore, the first components in the ordered pairs is called the domain of the relation and the set of second ordered pairs is called the range of the relation.

A relation from $X$ to $Y$ is a subset of $X \times Y$.

Suppose $R$ is a relation from $X$ to $Y$. Then $R$ is a set of ordered pairs where each first element comes from $X$ and each second element comes from $Y$. That is, for each pair $x \in X$ and $y \in Y$, exactly one of the following is true:

i. $(x, y) \in R$; we then say “$x$ is $R$-related to $y$”, written $x R y$.

ii. $(x, y) \notin R$; we then say “$x$ is not $R$-related to $y$”.

If $R$ is a relation from a set $X$ to itself, that is, if $R$ is a subset of $X^2 = X \times X$, then we say that $R$ is a relation on $X$.

**Example:** Find the domain and range of the relation linking the length of a woman’s femur to her height $\{(45.5, 65.5), (48.2, 68.0), (41.8, 62.2), (46.0, 66.0), (50.4, 70.0)\}$.

Solution:

Domain: $\{45.5, 48.2, 41.8, 46.0, 50.4\}$ (Set of first coordinates)

Range: $\{65.5, 68.0, 62.2, 66.0, 70.0\}$ (Set of second coordinates)

**Example:** Let $A = \{2, 3, 4\}$ and $B = \{3, 4, 5, 6, 7\}$. Define the relation $R$ by $aRb$ if and only if $a$ divides $b$. Find $R$, Domain of $R$, Range of $R$.

Solution

$R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}$

Domain $= \{2, 3, 4\}$

Range $= \{3, 4, 6\}$

**III. REPRESENTATIONS OF RELATIONS**

A relation may consist of a finite number of ordered pairs or an infinite number of ordered pairs. Furthermore, a relation may be defined by several different methods:

- **Arrow diagrams.** Venn diagrams and arrows can be used for representing relations between given sets. A relation may be defined by a correspondence. The corresponding ordered pairs are $\{(1, 2), (1, -4), (-3, 4), (3, 4)\}$.

- In the diagram an arrow from $x$ to $y$ means that $x$ is related to $y$. This kind of graph is called directed graph or digraph.

**Matrix of a Relation.** Another way of representing a relation $R$ from $A$ to $B$ is with a matrix. Its rows are labeled with the elements of $A$, and its columns are labeled with the elements of $B$. If $a \in A$ and $b \in B$ then we write 1 in row $a$ column $b$ if $aRb$, otherwise we write 0.

**For instance:** the relation $R = \{(a, 1), (b, 1), (c, 2), (c, 3)\}$ from $A = \{a, b, c, d\}$ to $B = \{1, 2, 3, 4\}$ has the following matrix:

$$
\begin{bmatrix}
1 & 2 & 3 & 4 \\
\downarrow & 0 & 0 & 0 \\
\downarrow & 1 & 0 & 0 & 0 \\
\downarrow & 0 & 1 & 1 & 0 \\
\downarrow & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

**IV. INVERSE RELATIONS**

Given a relation $R$ from $A$ to $B$, the inverse of $R$, denoted $R^{-1}$, is the relation from $B$ to $A$ defined as $bR^{-1} a$.
$R^{-1} = \{(b,a) : (a,b) \in R\}$

For instance, if $R$ is the relation “being a son or daughter of”, then $R^{-1}$ is the relation “being a parent of”.

**Example** Let $R = \{(1, 0), (2, 0), (2, 1), (3, 0), (3, 1), (3, 2)\}$ then

$R^{-1} = \{(0, 1), (0, 2), (1, 2), (0, 3), (1, 3), (2, 3)\}$

**Example** Let $R$ and $S$ be a relations between $A$ and $B$.

(i) Show that, if $R \subseteq S$ then $R^{-1} \subseteq S^{-1}$.

(ii) Prove that $\left[(R \cap S)\right]^{-1} = R^{-1} \cap S^{-1}$

**Proof (i)**

Let $(a,b) \in R^{-1}$.

$(a,b) \in R$, since $R \subseteq S$.

$\therefore (b,a) \in S^{-1}$ (definition of inverse relation)

$(b,a) \in S$, since $R \subseteq S$.

$\therefore (a,b) \in S^{-1}$ (definition of inverse relation)

$\therefore R^{-1} \subseteq S^{-1}$ (definition of subset)

**Proof (ii)**

Let $(a,b) \in (R \cap S)^{-1}$

$(a,b) \in R^{-1}$ and $(b,a) \in S^{-1}$ (definition of inverse)

$(a,b) \in R$ and $(b,a) \in S$ (definition of intersection)

$\therefore (a,b) \in (R \cap S)$ (definition of intersection)

$(a,b) \in (R \cap S)^{-1}$ (definition of inverse)

$R^{-1} \cap S^{-1} \subseteq (R \cap S)^{-1}$ (definition of intersection)

$\therefore$ from (1) and (2) we have

$(R \cap S)^{-1} = R^{-1} \cap S^{-1}$ (definition of subset)

V. **COMPOSITION RELATIONS**

Let $A$, $B$ and $C$ be three sets.

Given a relation $R$ from $A$ to $B$ and a relation $S$ from $B$ to $C$, then the composition $R \circ S$ of relations $R$ and $S$ is a relation from $A$ to $C$ defined by:

$R \circ S = \{(a,c) : (a,b) \in R$ and $(b,c) \in S$ for some $b \in B\}.$

For instance, if $R$ is the relation “to be the father of,” and $S$ is the relation “to be married to”, then $R \circ S$ is the relation “to be the father-in-law of.”

**Example** Let $R = \{(1, 2), (1, 6), (2, 4), (3, 4), (3, 6), (3, 8)\}$

$S = \{(2, u), (4, s), (4, t), (6, t), (8, u)\}$. Find $R \circ S$.

Solution.

$R \circ S = \{(1, u),(1, t),(2, s),(2, t),(3, s),(3, t),(3, u)\}$

**Example** Let $A = \{0, 1, 2, 3, 4, 5, 6, 7\}$, suppose $R$ and $T$ are Two relations on $A$ such that :

$R = \{(x,y) : 2x+3y=15\}, T = \{(x,y) : 3x+2y \in A\}$

Write down $R \circ T$ and $R \circ T$ as a set of ordered pairs ?

Solution

$R = \{(0,5),(3,3),(6,1)\}$

$T = \{(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(2,0)\}$

$R \circ T = \{(6,0),(6,1),(6,2)\}$

VI. **PROPERTIES OF BINARY RELATIONS**

A binary relation $R$ on a set $A$ is called:

1. **Reflexive** if for all $x \in A$, $xRx$. For instance on $Z$ the relation “equal to” (=) is reflexive.

**Example** (2.8) Let $A = \{a, b, c, d\}$ and $R$ be defined as follows:

$R = \{(a, a), (a, c), (b, a), (b, b), (c, c), (d, c), (d, d)\}$. 

**Proof**

Let $(a,b) \in R^{-1}$.

$(a,b) \in R$, since $R \subseteq S$.

$\therefore (b,a) \in S^{-1}$ (definition of inverse relation)

$(b,a) \in S$, since $R \subseteq S$.

$\therefore (a,b) \in S^{-1}$ (definition of inverse relation)

$\therefore R^{-1} \subseteq S^{-1}$ (definition of subset)
R is a reflexive relation.

**Example** Let R be a relation on a set then if R is reflexive then R⁻¹ is reflexive

**Proof**
Let (a,a)∈R ∀ a∈A
⇒ (a,a)∈R⁻¹ ∀ a∈A
⇒ R⁻¹ is reflexive

2. **Transitive** if for all x,y,z ∈ A, xRy and y Rz implies xRz. For instance equality (=) and inequality (<) on Z are transitive relations.

**Example** Let A = {a,b,c,d} and R be defined as follows:
R = {(a,b),(a,c),(b,d),(a,d),(b,c),(d,c)}. Here R is transitive relation on A.

3. **Symmetric** if for all x, y ∈ A, xRy implies y Rx. For instance on Z, equality (=) is symmetric, but strict inequality (<) is not.

**Example i.** Let A = {a,b,c,d} and R = {(a,a),(b,c),(c,b),(d,d)}.
Show that R is symmetric.
ii. Let R be the set of real numbers and R be the relation a R b if and only if a < b. Show that R is not symmetric.
Solution.
bRc and cRb so R is symmetric.
2 < 4 but 4 ≠ 2.

**Example** let R be a relation on a set A then R is symmetric iff R=R⁻¹

**Proof**
Assume R is a symmetric
let (a,b)∈R→ (b,a)∈R
(a,b)∈R⁻¹ and (b,a)∈R⁻¹
⇒ R=R⁻¹
Assume R=R⁻¹
let (a,b)∈R→(b,a)∈R⁻¹
(b,a)∈R since R=R⁻¹
⇒ R is symmetric

4. **Antisymmetric** if for all x, y ∈ A, xRy and y Rx implies x = y. For instance, non-strict inequality (≤) on Z is antisymmetric.

**Example** Let A = {a,b,c,d} and R be defined as:
R = {(a,b),(b,a),(a,c),(c,d),(d,b)}. R is not symmetric, as a R c but cRa. R is not anti-symmetric, because a R b and b R a, but a ≠ b.

5. **R is irreflexive** if, for every a∈A , (a,a)∉R

Example Let A = {a, b, c, d} and R be defined as follows:
R = {(a, a), (a, c), (b, a), (b, d), (c, c), (d, c), (d, d)}.
Here R is neither reflexive nor irreflexive relation as b is not related to itself and a, c, d are related to themselves.

Example let R be a relation on a set A then R is reflexive iff R^c is irreflexive

**Proof**
Let (a,a)∈R ∀ a∈A
⇒(a,a)∈R^c ∀ a∈A definition of complement
⇒ R^c is irreflexive definition of irreflexive
VII. PARTIAL ORDERS

Definition Let R be a binary relation on a nonempty set X. R is a partial ordering if R is a reflexive, transitive and antisymmetric relation. For example, the relation < is not a partial ordering, since it is transitive and antisymmetric but is not reflexive.

Example Let A = {1, 2, 3, 4, 6, 9} and relation R defined on A be “a divides b”. Is R a partial ordering relation on A?

Solution
First we list all ordered pairs of R as follows:

\[ R = \{(1,1),(1,2),(1,3),(1,4),(1,6),(1,9),(2,2),(2,4),(2,6),(3,3),(3,6),(3,9),(4,4),(6,6),(9,9)\} \]

\[ (1,1),(2,2),(3,3),(4,4),(6,6) \text{ and } (9,9) \in R \]

\[ \therefore \text{R is reflexive} \]

\[ (1,2),(1,3),(1,4),(1,6),(1,9),(2,4),(2,6),(3,6) \text{ and } (3,9) \in R \]

\[ \text{but } (2,1),(3,1),(4,1),(6,1),(9,1),(4,2),(6,2),(6,3) \text{ and } (9,3) \notin R \]

\[ \therefore \text{R is antisymmetric} \]

\[ (1,2),(2,4) \in R \text{ and } (1,4) \in R \]

\[ (1,3),(3,6) \in R \text{ and } (1,6) \in R \]

\[ (1,3),(3,9) \in R \text{ and } (1,9) \in R \]

\[ \therefore \text{R is transitive} \]

So R is a partial ordering relation.

VIII. EQUIVALENCE RELATIONS

An equivalence relation on a set A is a binary relation on A with the following properties:

Reflexive: for all \( x \in A \), \( x \; R \; x \).

Symmetric: if \( x \; R \; y \) then \( y \; R \; x \).

Transitive: if \( x \; R \; y \) and \( y \; R \; z \) then \( x \; R \; z \).

For instance, on \( \mathbb{Z} \), the equality (=) is an equivalence relation.

Another example, also on \( \mathbb{Z} \), is the following: \( x \; R \; y \) (mod 2) (“\( x \) is congruent to \( y \) modulo 2”) if \( x - y \) is even. For instance, 6 - 2 (mod 2) because 6 - 2 = 4 is even, but 7 ≠ 4 (mod 2), because 7 - 4 = 3 is not even. Congruence modulo 2 is in fact an equivalence relation:

Reflexive: for every integer \( x \), \( x - x = 0 \) is indeed even, so \( x \equiv x \) (mod 2).

Symmetric: if \( x \equiv y \) (mod 2) then \( x - y = t \) is even, but \( y - x = -t \) is also even, hence \( y \equiv x \) (mod 2).

Transitive: assume \( x \equiv y \) (mod 2) and \( y \equiv z \) (mod 2). Then \( x - y = t \) and \( y - z = u \) are even. From here, \( x - z = (x - y) + (y - z) = t + u \) is also even, hence \( x \equiv z \) (mod 2).

Equivalence Classes, Quotient Set and Partitions

Given an equivalence relation R on a set A, and an element \( x \in A \), the set of elements of related to \( x \) are called the equivalence class of \( x \), represented \([x] = \{y \in A \mid y \; R \; x\}\). The collection of equivalence classes, represented \( A/R = \{[x] \mid x \in A\}\), is called quotient set of A by R.

One of the main properties of an equivalence relation on a set A is that the quotient set, i.e. the collection of equivalence classes, is a partition of A. Recall that a partition of a set A is a collection of non-empty subsets \( A_1, A_2, A_3, \ldots \) of A which are pairwise disjoint and whose union equals A:

\[ 1. A_i \cap A_j = \emptyset \quad \text{for } i \neq j \]

\[ 2. \bigcup_{n} A_{n} = A \]

**Theorem** Let R be an equivalence relation on a set A. Then \( A/R \) is a partition of A. Specifically:

For each \( a \) in A, we have \( a \in [a] \).

\([a] = [b] \) if and only if \((a, b) \in R \).

If \([a] \neq [b] \), then \([a] \) and \([b] \) are disjoint.

Conversely, given a partition \( \{A_i\} \) of the set A, there is an equivalence relation R on A such that the sets \( A_i \) are the equivalence classes.

**Proof**

(i) Since R is reflexive, \((a, a) \in R\) for every \( a \in A \) and therefore \( a \in [a] \).
(ii) Suppose \((a, b) \in R\). We want to show that \([a] = [b]\).
Let \(x \in [b]\); then \((b, x) \in R\). But by hypothesis
\((a, a) \in R\) and so, by transitivity, \((a, x) \in R\). Accordingly \(x \in [a]\).
Thus \([b] \subseteq [a]\). To prove that \([a] \subseteq [b]\) we
observe that \((a, b) \in R\) implies, by symmetry, that \((b, a) \in R\).
Then, by a similar argument, we obtain \([a] \subseteq [b]\). Consequently, \([a] = [b]\).
On the other hand, if \([a] = [b]\), then, by (i), \(b \in [b] = [a]\); hence \((a, b) \in R\).

(iii) We prove the equivalent contrapositive statement:
If \([a] \cap [b] \neq \emptyset\) then \([a] = [b]\)
If \([a] \cap [b] \neq \emptyset\), then there exists an element \(x \in A\) with \(x \in [a] \cap [b]\).
Hence \((a, x) \in R\) and \((b, x) \in R\). By symmetry, \((x, b) \in R\)
and by transitivity, \((a, b) \in R\).
Consequently by (ii), \([a] = [b]\).

Example let \(A = \{1, 2, \ldots, 8\}\). Let \(R\) be the equivalence relation defined by \(x \equiv y \mod (4)\)
Write \(R\) as a set of ordered pairs
Find the partition of \(A\) induced by \(R\).

Solution
\[ R = \{(1, 1), (1, 5), (2, 2), (2, 6), (3, 3), (3, 7), (4, 4), (4, 8), (5, 1), (5, 5), (6, 2), (6, 6), (7, 3), (7, 7), (8, 4), (8, 8)\} \]
\([1] = \{1, 5\}\)
We pick an element which does not belong to \([1]\), say \(2\). Those elements related to \(2\) are \(2\) and \(6\), hence
\([2] = \{2, 6\}\)
\([3] = \{3, 7\}\)
The only element which does not belong to \([1]\), \([2]\) or \([3]\) is \(4\). The only element related to \(4\) is \(4\). Thus
\([4] = \{4, 8\}\)
Accordingly, the following is the partition of \(A\) induced by \(R\):
\[ A/R = \{[1], [2], [3], [4]\} \]

IX. FUNCTIONS
A special type of relation called a function.

Definition: Given a relation in \(x\) and \(y\), we say “\(y\) is a function of \(x\)” if for every element \(x\) in the domain, there corresponds exactly one element \(y\) in the range.

Note that the definition of a function requires that a relation must be satisfying two conditions in order to qualify as a function:
The first condition is that every \(x \in X\) must be related to \(y \in Y\) that is the domain of \(f\) must be \(X\) and not merely a subset of \(X\)
The second requirement of uniqueness can be expressed as:
\((x, y) \in f\) and \((x, z) \in f \Rightarrow y = z\)

Sometimes we represent the function with a diagram like this: \(f: A \rightarrow B\) or \(A \rightarrow B\)
For instance, the following represents the function from \(Z\) to \(Z\) defined by \(f(x) = 2x + 1\)

The element \(y = f(x)\) is called the image of \(x\), and \(x\) is a pre image of \(y\).

Remark: Functions are sometimes also called mappings or transformations.

To understand the difference between a relation that is a function and a relation that is not a function.
Example Determine which of the relations define \(y\) as a function of \(x\).
Solution

This relation is defined by the set of ordered pairs \{(1, -2), (2, 3), (3, 1)\}

Notice that for each x in the domain there is only one corresponding y in the range. Therefore, this relation is a function.

When x=1, there is only one possibility for y: y=-2
When x=2, there is only one possibility for y: y=3
When x=3, there is only one possibility for y: y=1

This relation is defined by the set of ordered pairs

When x=1, there are two possible range elements: y=2 and y=3

Therefore, this relation is not a function.

This relation is defined by the set of ordered pairs \{(1, 6), (2, 6), (3, 6)\}

When x=1, there is only one possibility for y: y=6
When x=2, there is only one possibility for y: y=6
When x=3, there is only one possibility for y: y=6

Because each value of x in the domain has only one corresponding y value, this relation is a function.

Remark: Vertical Line Test

A relation that is not a function has at least one domain element x paired with more than one range value y. For example, the ordered pairs (4, 2) and (4, -2) do not constitute a function because two different y-values correspond to the same x. These two points are aligned vertically in the xy-plane, and a vertical line drawn through one point also intersects the other point. Thus, if a vertical line drawn through a graph of a relation intersects the graph in more than one point, the relation cannot be a function.

This idea is stated formally as the vertical line test.

Example: Use the vertical line test to determine whether the following relations define y as a function of x
A function must have only one corresponding y-value in the range. In mathematics, functions are often given by rules or equations to define the relationship between two or more variables. For example, the equation $y=3x$ defines the set of ordered pairs such that the y-value is 3 times the x-value.

When a function is defined by an equation, we often use function notation. For example, the equation $y=3x$ can be written in function notation as $f(x)=3x$ where $f$ is the name of the function, $x$ is an input value from the domain of the function, and $f(x)$ is the function value (or y-value) corresponding to $x$.

The notation $f(x)$ is read as “$f$ of $x$” or “the value of the function $f$ at $x$.”

A function may be evaluated at different values of $x$ by substituting $x$-values from the domain into the function. For example, to evaluate the function defined by $f(x)=3x$ at $x=4$, substitute $x=4$ into the function:

$$f(4) = 3(4) = 12$$

Example Given the function defined by $f(x)=2x-1$.

Find the function values i. $f(0)$ ii. $f(1)$ iii. $f(-1)$ iv. $f(2)$

Solution

$f(0)=2(0)-1$ 
$= -1$ We say, “$f$ of 0 is -1.” This is equivalent to the ordered pair (0,-1) 

$f(1)=2(1)-1$ 
$= 1$ We say, “$f$ of 1 is 1.” This is equivalent to the ordered pair (1,1) 

$f(-1)=2(-1)-1$ 
$= -3$ We say, “$f$ of -1 is -3.” This is equivalent to the ordered pair (-1,-3) 

$f(2)=2(2)-1$ 
$=4-1=3$ We say, “$f$ of 2 is 3.” This is equivalent to the ordered pair (2,3)
X. DOMAIN AND RANGE OF FUNCTIONS

You begin to format your paper, first write and save the content as a separate text file. Keep your text and graphic files separate until after the text has been formatted and styled. Do not use hard tabs, and limit use of hard returns to only one return at the end of a paragraph. Do not add any kind of pagination anywhere in the paper. Do not number text heads—the template will do that for you.

Finally, complete content and organizational editing before formatting. Please take note of the following items when proofreading spelling and grammar. The function will be undefined when the denominator is zero, that is, when $2x-1=0$

$$2x=1$$

$x = 1/2$  The value $x = 1/2$ must be excluded from the domain.

Interval notation: $(-\infty, 1/2) \cup (1/2, \infty)$

b. The quantity is greater than or equal to 0 for all real numbers $x$, and the number 9 is positive. Therefore, the sum must be positive for all real numbers $x$. The denominator of $g(x) = (x-4)/(x^2+9)$ will never be zero; the domain is the set of all real numbers. Interval notation: $(-\infty, +\infty)$

Graphs of Basic Functions

We can associate a set of pairs in $A \times B$ to each function from $A$ to $B$. This set of pairs is called the graph of the function and is often displayed pictorially to aid in understanding the behavior of the function.

Definition  Let $f$ be a function from the set $A$ to the set $B$. The graph of the function $f$ is the set of ordered pairs

$$\{(a, b) \mid a \in A \text{ and } f(a) = b\}.$$  

From the definition, the graph of a function $f$ from $A$ to $B$ is the subset of $A \times B$ containing the ordered pairs with the second entry equal to the element of $B$ assigned by $f$ to the first entry.

Also, note that the graph of a function $f$ from $A$ to $B$ is the same as the relation from $A$ to $B$ determined by the function $f$.

To determine the shapes of the basic functions, we can plot several points to establish the pattern of the graph. Analyzing the equation itself may also provide insight to the domain, range, and shape of the function.

Example Graph the functions defined by $f(x) = x^2$

Solution

The domain of the function given by $f(x) = x^2$ or equivalently $y = x^2$ is all real numbers.

To graph the function, choose arbitrary values of $x$ within the domain of the function. Be sure to choose values of $x$ that are positive and values that are negative to determine the behavior of the function to the right and left of the origin. The function values are equated to the square of $x$, so $f(x)$ will always be greater than or equal to zero. Hence, the $y$-coordinates on the graph will never be negative. The range of the function is $\{y \mid y$ is a real number and $y \geq 0\}$.

The arrows on each branch of the graph imply that the pattern continues indefinitely.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x) = x^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>9</td>
</tr>
<tr>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
</tbody>
</table>

Types of Functions

One-to-One or Injective: A function $f: A \rightarrow B$ is called one-to-one or injective if each element of $B$ is the image of at most one element of $A$.

$\forall x, x' \in A, f(x) = f(x') \implies x = x'$ For instance, $f(x) = 2x$ from $Z$ to $Z$ is injective.
For instance, $f(x) = 2x$ from $\mathbb{Z}$ to $\mathbb{Z}$ is injective.

3. One-To-One Correspondence or Bijective: A function $f : A \rightarrow B$ is said to be a one-to-one correspondence, or bijective, or bijection, if it is one-to-one and onto.

**Inverse Function**

If $f : A \rightarrow B$ is a bijective function, its inverse is the function $f^{-1} : B \rightarrow A$ such that $f^{-1}(y) = x$ if and only if $f(x) = y$

A characteristic property of the inverse function is that $f^{-1} \circ f = 1_{A}$ and $f \circ f^{-1} = B$
Example let \( f \) be the function from \{ a, b, c \} to \{ 1, 2, 3 \} such that \( f( a) = 2, f( b) = 3, \) and \( f( c) = 1. \) Is \( f \) invertible, and if it is, what is its inverse?

**Solution**

The function \( f \) is invertible because it is a one-to-one correspondence. The inverse function \( f^{-1} \) reverses the correspondence given by \( f, \) so \( f^{-1}(1) = c, f^{-1}(2) = a \) and \( f^{-1}(3) = b. \)

Example let \( f: \mathbb{Z} \rightarrow \mathbb{Z} \) be such that \( f(x) = x + 1. \) Is \( f \) invertible, and if it is, what is its inverse?

**Solution**

The function \( f \) has an inverse because it is a one-to-one correspondence, as follows:

To reverse the correspondence, suppose that \( y \) is the image of \( x, \) so that \( y = x + 1. \) Then, \( x = y - 1. \) This means that \( y - 1 \) is the unique element of \( \mathbb{Z} \) that is sent to \( y \) by \( f. \)

**Definition Identity function.** Given a set \( A, \) the function \( 1_A: A \rightarrow B \) defined by \( 1_A(x) = x \) for every \( x \) in \( A \) is called the identity function for \( A. \)

**Remark Geometrical Characterization of one-to-one and onto functions**

Consider now functions of the form \( f: \mathbb{R} \rightarrow \mathbb{R}. \) Since the graphs of such functions may be plotted in the Cartesian plane \( \mathbb{R}^2 \) and since functions may be identified with their graphs, we might wonder whether the concepts of being one-to-one and onto have some geometrical meaning. The answer is yes.

Specifically:

- \( f: \mathbb{R} \rightarrow \mathbb{R} \) is one-to-one if each horizontal line intersects the graph of \( f \) in at most one point.
- \( f: \mathbb{R} \rightarrow \mathbb{R} \) is an onto function if each horizontal line intersects the graph of \( f \) at one or more points.

Accordingly, if \( f \) is both one-to-one and onto, i.e. invertible, then each horizontal line will intersect the graph of \( f \) at exactly one point.

**Function Composition.** Given two functions \( f: A \rightarrow B \) and \( g: B \rightarrow C \) the composite function of \( f \) and \( g \) is the function \( g \circ f = A \rightarrow C \) defined by \( (g \circ f)(x) = g(f(x)) \) for every \( x \) in \( A: \)

In other words, \( g \circ f \) is the function that assigns to the element \( a \) of \( A \) the element assigned by \( g \) to \( f(a). \) That is, to find \( (g \circ f)(a) \) we first apply the function \( f \) to \( a \) to obtain \( f(a) \) and then we apply the function \( g \) to the result \( f(a) \) to obtain \( (gof)(a) = g(f(a)). \) Note that the composition \( g \circ f \) cannot be defined unless the range of \( f \) is a subset of the domain of \( g. \)

**Example** Let \( g \) be the function from the set \( \{ a, b, c \} \) to itself such that \( g(a) = b, g(b) = c, \) and \( g(c) = a. \) Let \( f \) be the function from the set \( \{ a, b, c \} \) to the set \( \{ 1, 2, 3 \} \) such that \( f(a) = 3, f(b) = 2, \) and \( f(c) = 1. \) What is the composition of \( f \) and \( g, \) and what is the composition of \( g \) and \( f? \)

**Solution:**

The composition \( f \circ g \) is defined by \( (f \circ g)(a) = f(g(a)) = f(b) = 2, \)
\( (f \circ g)(b) = f(g(b)) = f(c) = 1, \) and \( (f \circ g)(c) = f(g(c)) = f(a) = 3. \)

Note that \( g \circ f \) is not defined, because the range of \( f \) is not a subset of the domain of \( g. \)

**Example** Let \( f \) and \( g \) be the functions from the set of integers to the set of integers defined by \( f(x) = 2x + 3 \) and \( g(x) = 3x + 2. \) What is the composition of \( f \) and \( g? \) What is the composition of \( g \) and \( f? \)

**Solution:**

Both the compositions \( f \circ g \) and \( g \circ f \) are defined.

Moreover,
\( (f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7 \)
and
\( (g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11. \)

Example Let functions \( f \) and \( g \) defined as follows:

\( f(x) = 2x + 1 \) and \( g(x) = x^2 - 2 \) respectively, find \( g(f(4)) \) and \( f(g(4))? \)

**Solution:**

\( (gof)(4) = g(f(4)) \)
Note that even though \( f \circ g \) and \( g \circ f \) are defined for the functions \( f \) and \( g \), \( f \circ g \) and \( g \circ f \) are not equal. In other words, the commutative law does not hold for the composition of functions.

Some properties of function composition are the following:

1. If \( f : A \rightarrow B \) is a function from \( A \) to \( B \), we have that \( f \circ 1_A = 1_B \circ f = f \).

Function composition is associative, i.e., given three functions \( A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \), we have that \( h \circ (g \circ f) = (h \circ g) \circ f \).

Example Let functions \( f, g \) and \( h \) defined as follows:

\[
\begin{align*}
  f(x) &= 3x & f: N \rightarrow N \\
  g(x) &= 2x^2 & g: N \rightarrow N \\
  h(x) &= 5x & h: N \rightarrow N
\end{align*}
\]

Where \( N \) is Positive integers. Find \( (h \circ g \circ f)(x) \).

Solution

\[
(h \circ g \circ f)(x) = (h \circ g \circ f)(x) = h(g(f(x))) = h(3x(2x^2)) = h(18x^2) = 90x^2
\]

When the composition of a function and its inverse is formed, in either order, an identity function is obtained. To see this, suppose that \( f \) is a one-to-one correspondence from the set \( A \) to the set \( B \). Then the inverse function \( f^{-1}(\cdot) \) exists and is a one-to-one correspondence from \( B \) to \( A \). The inverse function reverses the correspondence of the original function, so \( f^{-1}(b) = a \) when \( f(a) = b \), and \( f^{-1}(b) = a \) when \( f^{-1}(b) = a \). Hence,

\[
\begin{align*}
  f^{-1} \circ f)(a) &= f^{-1}(f(a)) = f^{-1}(b) = a, \\
  (f \circ f^{-1})(b) &= f(f^{-1}(b)) = f(a) = b.
\end{align*}
\]

Consequently \( f^{-1} \circ f = 1_A \) and \( f \circ f^{-1} = 1_B \), where \( 1_A \) and \( 1_B \) are the identity functions on the sets \( A \) and \( B \), respectively. That is, \( (f^{-1})^{-1} = f \).

DIFFERENCE BETWEEN RELATION AND FUNCTION

Using the example we can write the relation in set notation: \{ (apples, sweetness), (apples, tartness), (oranges, tartness), (bananas, sweetness) \}. The inverse relation, which we could describe as "fruits of a given flavor", is \{ (sweetness, apples), (sweetness, bananas), (tartness, apples), (tartness, oranges) \}. One important kind of relation is the function. A function is a relation that has exactly one output for every possible input in the domain. (The domain does not necessarily have to include all possible objects of a given type. In fact, we sometimes intentionally use a restricted domain in order to satisfy some desirable property.) The relations discussed above (flavors of fruits and fruits of a given flavor) are not functions: the first has two possible outputs for the input "apples" (sweetness and tartness); and the second has two outputs for both "sweetness" (apples and bananas) and "tartness" (apples and oranges). The main reason for not allowing multiple outputs with the same input is that it lets us apply the same function to different forms of the same thing without changing their equivalence. That is, if \( f \) is a function with a (or b) in its domain, then \( a = b \) implies that \( f(a) = f(b) \). For example, \( z - 3 = 5 \) implies that \( z = 8 \) because \( f(x) = x + 3 \) is a function defined for all numbers \( x \). The converse, that \( f(a) = f(b) \) implies \( a = b \), is not always true. When it is never more than one input \( x \) for a certain output \( y = f(x) \). This is the same as the definition of function, but with the roles of \( X \) and \( Y \) interchanged; so it means the inverse relation \( f^{-1}(\cdot) \) must also be a function. In general—regardless of whether or not the original relation was a function—the inverse relation will sometimes be a function, and sometimes not.
CONCLUSION
In this paper relation and function will be the focus of most of the rest of algebra, as well as pre-calculus and calculus. It is an important stepping stone to the rest of algebra. Relation and Function will be used to solve many different types of problems. However, one must first learn the basics—how to recognize a function, and how to determine its domain and function. It explains how to represent relation and function using both mapping diagram and graphs.

REFERENCES: