CHARACTERIZATION OF GENERALIZED CLASS OF \(\ast\)-BISIMPLE AMPLE \(\omega\)-SEMIGROUPS

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ABSTRACT

Generalized class of \(\ast\)-Bisimple Ample \(\omega\)-Semigroups whose certain \(\mathcal{H}^\ast\)-classes contain no regular elements were considered in [3]. Thus, the internal characterization of this class of semigroups rather involves the use of bisystems with certain structure mappings and a binary multiplication together with some closure axioms. However, these structural characterizations make this class of ample semigroups different from those studied in [2] and has no analogue to those studied in [9]. This paper is a follow up of [3], which further characterized this class of ample semigroups.

Key words: Cancellative monoids, abundant semigroups, adequate semigroups, ample semigroups, \(\omega\)-chain, \(\ast\)-bisimple semigroups, binary array of bisystems of cancellative monoids.

Throughout this paper \(S\) denotes a semigroup (except stated otherwise), terminologies are curled from [2] and [3].

1.0 Preliminaries

Definition 1.1
Let \(a, b \in S\), then Green’s \(\ast\)-relations are defined as follows:

\(\mathcal{L}^\ast = \{(a,b) \in S \times S; \forall x, y \in S^1. ax = ay \iff bx = by\}\)

\(\mathcal{R}^\ast = \{(a,b) \in S \times S; \forall x, y \in S^1. xa = ya \iff xb = yb\}\)

\(\mathcal{H}^\ast = \{(a,b) \in S \times S; (a,b) \in \mathcal{L}^\ast \cap \mathcal{R}^\ast\}\)

\(\mathcal{D}^\ast = \{(a,b) \in S \times S; (a,b) \in \mathcal{L}^\ast \cup \mathcal{R}^\ast\}\), that is; \(\exists c \in S; (a,c) \in \mathcal{R}^\ast, (c,b) \in \mathcal{L}^\ast \lor a\mathcal{R}^\ast c\mathcal{L}^\ast b\).

The following are well known results;

Lemma 1.2 (5)
\(\mathcal{R}^\ast(\mathcal{L}^\ast)\) is a left(right) congruence.

Lemma 1.3[7]
\(i. \quad (ax, a) \in \mathcal{R}\)

\(ii. \quad (a,xa) \in \mathcal{L}\)

\(iii. \quad (ax, xa) \in \mathcal{D}\)

Proof
\(i (ax, a) \in \mathcal{R} \Rightarrow \exists s, t \in S^1: axs = a, at = ax\)

\(axs = a \Rightarrow axst = at \Rightarrow axst = ax \Rightarrow at = ax.

ii is a dual of i, iii is an immediate consequence of i and ii

Lemma 1.4 [5]
Let \(a \in S, e \in E_S\). Then the following holds;

(i) \((e, a) \in \mathcal{L}^\ast\).

(ii) \(ae = a\) and for all \(x, y \in S^1, axy = ay \Rightarrow ex = ey\).

Dually

Let \(a \in S, e \in E_S\). Then;

(i) \((a, e) \in \mathcal{R}^\ast\).

(ii) \(ea = a\) and for all \(x, y \in S^1, xay = ya \Rightarrow xe = ye\).

Lemma 1.5
Let \(S\) be an adequate semigroup with semilattice of idempotents \(E\), then \(\forall a, b \in S\),

\(i. \quad (a,b) \in \mathcal{R}^\ast\) if and only if \(a^+ = b^+\); \((a,b) \in \mathcal{L}^\ast\) if and only if \(a^+ = b^+\).

\(ii. \quad (ab)^\ast = (a^\ast b)^\ast\) and \((ab)^\ast = (ab)^\ast\)^+.

\(iii. \quad aa^\ast = a = a^\ast a\).
2.0 Characterization of a Generalized $*$-Simple Ample $\omega$-Semigroups

Definition 2.0
Let $S$ be a semigroup, and let $a, b \in S$, the relation $\overline{D}$ on $S$ is defined by:

$$a \overline{D} b \iff a^* \overline{D} b^*, \ a^* \overline{D} b^*.$$ 

Lemma 2.1
On $S$, the relation $\overline{D}$ satisfies the inclusion $\mathcal{D}(S) \subseteq \overline{D}(S) \subseteq D^*(S)$.

Proof
Observe that if $(a^*, b^*) \in \overline{D}(S), (a^*, b^*) \in \overline{D}(S) \Rightarrow (a, b) \in D^*(S) \Rightarrow \overline{D}(S) \subseteq D^*(S)$.

We now show that $\mathcal{D}(S) \subseteq \overline{D}(S)$.

Observe that if $(a, b) \in \mathcal{D}(S)$, then there exists $c \in S$, such that $aRc \subseteq b$ with $a^* \in L_c^*$ and $b^* \in R_c^*$ and so $(c, a^*) \in L$. Now because $(c, b) \in R$ then there exists $x, y \in S^1$ such that $cx = by = c$ thus $cxy = c$ and since $c^* = a^*$ it follows that $a'xy = a'$ so that $(a'x, a^*) \in R$ showing the regularity of $a'x$. But $(c, a^*) \in L^*$ and $(b, b^*) \in L^*$ and so, for any $s, t \in S^1, a's = a't \Rightarrow cxs = cxt$ and $bs = bt \Rightarrow b's = b't$ and then $(a'x, b^*) \in L$ and therefore $a'Ra'xLb^* \Rightarrow (a^*, b^*) \in \overline{D}(S)$. Similarly, $(a^*, b^*) \in \overline{D}(S)$.

Lemma 2.2
On $S$, the following are equivalent;

i. $\overline{D}(S) = D^*(S)$

ii. Every non-empty $H^*$-class of $S$ contains some regular element.

Proof
Suppose i holds, then from lemmas 1.3 and 2.1 above $(a^*, b^*) \in \overline{D}(S) \Rightarrow (a^*, a'x) \in R, (a^*, b^*) \in L$. For simplicity let $e = a^*, f = b^*$ and $v = a'x$. Thus, by lemma 1.4, $(e, v) \in R \Rightarrow ev = v$. Similarly, $(v, f) \in L \Rightarrow vf = v$. Again, $(e, v) \in R \Rightarrow \exists s, t \in S^1$ such that $vs = e$ and $te = v$. Now let $v' = vse$, so that $vv'v = v(fse)v = (vf)s(ev) = vsv = ev = v$ and so $v' \in V(v)$, where $V(v) = \{v \in S: vv'v = v \text{ and } v'vv' = v''\}$. Also, $v'' = v(fse)v = (vf)se = use = ee = e$. But also, since $(v, f) \in L$ then there exists $t \in S^1$ such that $tv = f$, thus, $v'v = (fse)v = fsev = fsv = tvsv = tev = tv = f$. It now follows easily that $v' \in L_e \cap R_f = L_u \cap R_u = H_u$ for some $u \in H^*$. But $H^* \subseteq H^*$.

ii $\Rightarrow$ i is straightforward since there exists an inverse $v'$ of $v$ in $L_e \cap R_f$ such that $e = vv', f = v'v$ and so $(e, v) \in R$ and $(v, f) \in L \Rightarrow (e, f) \in D \Rightarrow (a^*, b^*) \in D \subseteq \overline{D}(S) = D^*(S)$.

The following are results from [3]

Theorem 2.3 [3]
$S = S(B, d, \theta, \varphi)$ is a semigroup.

Lemma 2.4 [3]
The idempotents of $S$ are of the form $(m, e_m, m)$.

Lemma 2.5 [3]
Let $a = (m, x, n) \in S$, and let $x \in M_m$, where $x$ is a unit, then the inverse of $a$, is of the form $a^{-1} = (n, y, m)$, where $y = x^{-1}$ and $m = n(\text{mod} d)$.

Lemma 2.6 [3]
Let $a = (m, x, n), f_n = (n, e_m, n), f_m = (m, e_m, m) \in S$. Then for all $u = (h, y, k), v = (f, z, g) \in S$, then:

i. $a^L f_n$ and

ii. $a^R f_m$.

Define the set of idempotents of $S$ as $E(S) = \{f_m: m \geq 0\}$.

Suppose that $f_m = (m, e_m, m)$, $f_n = (n, e_m, n), \in E(S), m, n \in N$. Observe that

\[f_m f_n = (m, e_m, m)(n, e_m, n) = (1, e_0 e_t^{m - n}{e_0 t})\]

Where $t = \max(m_n, n), t' = \max(m, n_d)$. Now observe that if $t = m, t' = m_d$ then we have;

\[f_m f_n = (m, e_0 e_t^{m - n}{e_0 t}, m) = (m, e_0 e_m, m)\]

Similarly, if $t = n$, then $t' = n_d$, then * becomes:

\[f_m f_n = (n, e_m e_t^{n - m}{e_0 n}, n) = (m, e_m e_0, m), \text{ and if } t = m, \text{ then } f_m f_n = (m, e_m, m) = f_m\]

Thus:
\[ f_{mn} = \{(m, e_m e_0, m) = (m, e_m, m), t = m > n \] 

Therefore, we can now define a partial order on \( E(S) \) as follows:

\[ f_m \leq f_n \Rightarrow f_{mn} = f_n f_m = f_m \] if \( m \geq n \)

Thus, we observe the following:

1. \( f_m \leq f_m \)
2. \( f_m \leq f_n \)
3. \( f_m \leq f_n, f_n \leq f_i \Rightarrow f_{mn} = f_m \) and \( f_n \leq f_i \Rightarrow f_{fn} = f_n \). Thus \( f_m(f_{fn}) = (f_{mn}) f_i \)

Thus, the relation \( \leq \) is an equivalence relation. \( \square \)

**fact**

If \( f_0 \) is an idempotent in \( S \) we have:

\[ f_0 = (0, e, 0) \geq f_1 = (1, e, 1) \geq f_2 = (2, e, 2) \ldots \geq f_d = (d, e, d) \text{ and in general,} \]

\[ f_d = (d, e, d) \geq f_{d+1} = (d + 1, e, d + 1) \geq \ldots \geq f_{2d-1} = (2d - 1, e, 2d - 1) \geq \ldots \]

Thus, for \( f_m, f_n \in E(S), f_m \leq f_n \) if and only if \( n \leq m \) and so \( E(S) = \{f_m | m \geq 0\} \) forms an \( \omega \)-chain in \( S \) with respect to the partial order. Evidently, \( f_0 \in S \) is an identity.

Let \( a = (m, x, n), b = (p, y, q) \) so that \( af_n = (m, x, n)(n, e, n) = (m, x, n) \),

\[ bf_n = (p, y, q)(n, e, n) = (p - q + \omega, x\theta^{\omega-a}e\varphi \omega^{-a}, w). \]

But \( bf_n = b \Rightarrow q = w = \max(q, n) \).

Similarly, \( af_n = (m, x, n)(q, e, q) = (m - n + \omega, x\theta^{\omega-a}e\varphi \omega^{-a}, v) \) and so

\[ af_n = a \Rightarrow n = w = \max(n, q). \]

That is \( af_n = a = (q \leq n) \) and \( bf_n = b \Rightarrow n \leq q \).

However, \( af_n = a \Rightarrow bf_n = b \Rightarrow n = q \).

In general, \( au = av \) then from lemma 2.6 above, we have that

\[ x\varphi^{\omega-a}y\varphi^{\omega-a} = x\varphi^{\omega-a}z\varphi^{\omega-a} \Rightarrow e_n\theta^{\omega-a}y\varphi^{\omega-a} = e_n\theta^{\omega-a}z\varphi^{\omega-a} \]

(Particularly when \( x = e_n \), we have that:

\[ a \) e_ny = e_nz \), which evidently gives that \( y = z \)

b) \( e_n\varphi^{\omega-a} = e_nz\varphi^{\omega-a} \)

Thus, in b), recall that \( y\varphi^{\omega-a} \subseteq M_{p-\theta, 0} = M_{p-\theta, 0} \) and so if we let \( y' = y\varphi^{\omega-a} \), then

\[ y', y\varphi^{\omega-a} = (y'e_n)\varphi^{\omega-a} = y'(e_nz)\varphi^{\omega-a} = (y'e_n)z\varphi^{\omega-a} = y'z\varphi^{\omega-a} \] so that \( (p, y, q)(h, y, k) = (p, y, q)(f, z, k) \) and we have proved

**Lemma 2.7**

\[ a\mathcal{L} f_n \text{ and } a\mathcal{R} f_m, \text{ then every } \mathcal{L} \text{ and every } \mathcal{R} \text{ class of } S \text{ contains an idempotent}. \] \( \square \)

**Lemma 2.8**

\( S \) is abundant.

**Proof**

This follows immediately from lemma 2.6 above since:

\[ a\mathcal{L} f_n \text{ and } a\mathcal{R} f_m, \text{ then every } \mathcal{L} \text{ and every } \mathcal{R} \text{ class of } S \text{ contains an idempotent}. \] \( \square \)

Now, suppose \( f_n = (n, e_n, n), f_m = (m, e_m, m) \in S \), then observe that

\[ f_{mn} = f_m f_n = (n, e_n, n)(m, e_m, m) = (t, e_m e_n, t) = m \geq n \]

Thus, we have proved

**Lemma 2.9**

\( S \) is adequate

Let \( a = (m, x, n) \in S \) and let \( f_m = (m, e_m, m), f_n = (n, e_n, n), f_p = (p, e_p, p) \) be an idempotents in \( S \). Then observe that:

\[ af_n = (m, x, n)(n, e_n, n) = (m, x e_n, n) = (m, e_m, m)(m, x, n) = f_m a. \]

\[ f_p a = (m, e_m, p)(m, e_m, m) = (v, e_p \theta^{\omega-a} x \varphi^{\omega-\omega} m - n + v) \] and if \( m > p \), then

\[ f_p a = (m, e_m \theta^{\omega-a} x, m) = (m, e_m, m)(m, x, n) = f_p (f_m a). \]

\[ f_{f_m} a = (m, e_m, p)(m, e_m, m)(m, x, n) = (m, e_p \theta^{\omega-a} m - n, e_m, m)(m, x, n) \]

\[ = (m, e_p \theta^{\omega-a} m - n, e_m, m)(x, n) = (m, e_p \theta^{\omega-a} m - n, e_m, m)(x, n) \]

\[ = (m, e_p \theta^{\omega-a} m - n, e_m, m)(x, n) = (m, e_p \theta^{\omega-a} m - n, e_m, m)(x, n) \]

and so

\[ f_p a = f_p (f_m a) = (f_{f_m} a) a = f_n a. \]
However, if $p > m$, then putting $r = n - m + p$, we have:

$$f_p a = (p, e, p) (m, x, n) = (p, e, x, p^{n-nd}, n - m + p) = (p, e, x, p^{r-nd}, r)$$

But $(p, e, x, p^{r-nd}, r) = (m, x, n) (r, e, r) = af_r = a (f_p a)^*$, where $(f_p a)^* = (r, e, r)$.

Thus, evidently, $f_p a = af_r = a (f_p a)^*$.

Similarly, $a f_p = (m, x, n) (p, e, p, p) = (m - n + v, x, u p^{n-nd} e, p^{n-nd} u, v) = (m, x, n) = [(m, x, n) (n, e, n)] (p, e, p, p)$

Thus $f_p = (r, x, p^{n-nd} e, p, p) = (r, x, r^{n-nd} e p, n - m + r) = (r, e, r) (m, x, n) = f_p a$ since $r = m - n + p = n - m + p$.

Thus, for whatever case, evidently, $af_p = f_a a = (af_p)^* a$, $(af_p)^* = (r, e, r)$, thus, along fact* and lemmas 2.8, 2.9, we have proved:

**Theorem 2.10**

$S = S(B, d, \theta, \varphi)$ is an ample $\omega$-semigroup.

Observe that for all $i, j \in d$, $B_{i,j} \neq \emptyset$, the $L^*$-class and $R^*$-class of $S$ respectively are of the form:

$L_{n,i}^* = \{ (m, x, n) : m, n \in N, x \in M_{i,n}, 0 \leq i \leq d - 1 \}$

$R_{m,j}^* = \{ (m, x, n) : m, n \in N, x \in M_{m,j}, 0 \leq j \leq d - 1 \}$

Thus each $H^*$-class of $S$ can be expressed as:

$H_{m,n}^* = \{ (m, x, n) : m, n \in N, x \in B_{m,n}^* \}$

Now observe that $H_{m,n}^* = \{ (m, x, n) : m, n \in N, x \in B_{m,n} \} \neq \emptyset$ since $B_{m,n} \neq \emptyset$.

Let $a = (m, x, n) \in H_{m,n}^*$ along with the fact that $(f_m a) \in R^*$ and $(a, f_n) \in L^*$ that is:

$f_m R^* a L^* f_n$ and so $(f_m, f_n) \in D^*$ and then together with theorem 2.10 above, we have proved:

**Theorem 2.11**

$S$ is a $*$-bimodal ample $\omega$-semigroup.

**Conclusion**

[3] constructed a generalized class of $*$-bimodal ample $\omega$-semigroups and characterized them as an extension of the binary array of bisystems of cancellative monoids. Here we present further characterization of this class of semigroups. Lemma 2.7 gives a condition under which the elements of $S$ are $R^* (L^*)$-related while lemmas 2.8 and 2.9 respectively showed that $S$ is abundant and adequate. Theorems 2.10 and 2.11 respectively showed that $S$ is ample $\omega$-semigroup and $*$-bimodal. In all, further characterizations of generalized class of $*$-bimodal ample $\omega$-semigroups have been established.

**References**