INFINITE SEMIGROUPS WHOSE NUMBER OF INDEPENDENT ELEMENTS IS LARGER THAN THE BASIS.

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Abstract - We investigate a certain characterisation for rank of a semigroup by Howie and Ribeiro (1999), to ascertain the relevance of the concept of independence. There are cases where the concept of independence fails to be useful for this purpose. One would expect the basis element to be the maximal independent subset of a given semigroup. However, we construct examples for infinite, commutative and non-commutative semigroups, where there exists finite basis and the number of independent elements is larger than the basis.

Key Words: Generating sets, Independent Sets, Rank, Cyclic Semigroup, Basis, Commutative Semigroup.

1.0. Introduction and Preliminaries

An algebraic structure has many elements and it is practical to have a small subsystem of the structure which facilitates the construction of all the elements using the operation(s) of the structure. Such subsystems are called generator systems. Our interest is to study such small subsystems especially as it concerns their sizes. In particular, the size $k$ of the smallest subset $U$ of $S$, in which every element of $S$ can be generated as the product of the elements of $U$ is what we call the rank of the semigroup.

The notion of ‘rank’ or ‘dimension’ belongs primarily to linear algebra. In linear Algebra, Rank can be defined for a finite vector space $V$ either as the size of a maximal linear independent subset or as the size of a minimal generating set of $V$, and it is an elementary result in linear algebra that these two cardinalities are equal.

However, in the case of a more general algebraic system like semigroup, the possible definitions of rank are not just in terms of maximal linear independent subsets or as the size of a minimal generating set because results show that both do not coincide in some cases unlike in the case of Vector Spaces. One example is the Brandt semigroup in which the size of maximal independent subset is different from the size of its minimal generating set.

Consequent upon this fact that the rank of a semigroup in terms of minimal generating set fails to coincide in some cases, with the rank in terms of maximal independent set, Howie and Ribeiro (1999) gave a definition for five (5) different types of rank (as we shall later see) to sufficiently characterise rank for all possible semigroups. This definition accommodates the idea of minimal generating set and that of maximal independent subset. The five ranks given in this definition do not all give the same value so that it is possible to have varying values as rank for a particular semigroup depending on which definition we choose to use. Apart from the above limitation of the concept of independence in characterising ranks in semigroups, our work further strengthens the argument that independence cannot be relied upon for characterising ranks as it is possible to have subsystems of a semigroup in which there exist independent subsets that are larger than the basis.

In the first section of the work, we define useful terms as well as give example to illustrate generating sets and independent sets. In section 2, we present the characterisation of rank to be studied. In Sections 3 and 4 we use concrete examples of infinite commutative and infinite non-commutative semigroups to show our claim.

1.1. Definition

A group is a set $G$ together with a binary operation $*$ such that

(i) $\exists e \in G$ such that $e \ast x = x = x \ast e, x \in G$
(ii) for any $x \in G$, $\exists x^# \in G$ such that $x \ast x^# = x^# \ast x = e \in G$
(iii) for $x, y, z \in G, x \ast (y \ast z) = (x \ast y) \ast z$
1.2. Definition
A semigroup is a set $S$ together with a binary operation $*$ (that is a function $*: S \times S \to S$) that satisfies the associative property:

$$a * (b * c) = (a * b) * c$$

holds.

1.3. Definition (Generating sets)
Let $X$ be a subset of a semigroup $S$, $X$ generates $S$ as a semigroup if every element of $S$ can be written as a product of elements of $X$.

1.4. Definition (Independent Sets of semigroup)
Let $S$ be a finite semigroup. A subset $U$ of $S$ is independent if, for every $u$ in $U$, the element $u$ does not belong to the subsemigroup $\langle U \setminus \{u\} \rangle$ generated by the remaining elements of $U$.

1.5. Definition (Primary definition of Rank of a semigroup)
The rank $r(S)$ of a semigroup $S$ is the size of the minimal generating set $U \subseteq S$.

1.6. Example
Consider the six elements semigroup $S$, which is a $2 \times 2$ real matrix under matrix multiplication,

\[
o = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad i = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad ab = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad ba = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\]

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Let $U = \{i, a, b\} \subseteq S$. We show that $U$ is a maximal independent subset:

If we take out $i$ from $U$, notice that $\{i\}$ is excluded from the subsemigroup generated by the remaining elements $U \setminus \{i\}$ of $S$. That is, $\{i\}$ is excluded from $\langle U \setminus \{i\} \rangle = \{o, a, b, ab, ba\}$.

Notice also that $\{a\}$ and $\{b\}$ are excluded from the subsemigroup $\langle U \setminus \{a\} \rangle = \{o, i, b\}$ and $\langle U \setminus \{b\} \rangle = \{o, i, a\}$ respectively. We conclude that the subset $U$ is independent.

Next, we show that $U$ is a maximal independent subset of the semigroup $S$.

Notice that $U = \{i, a, b\}$ becomes dependent if we include any of $o$ or $ab$ or $ba$ in $U$:
The subsemigroup generated by \{o, i, a, b\} \setminus \{o\} is \{o, i, a, b, ab, ba\}

The subsemigroup generated by \{ab, i, a, b\} \setminus \{ab\} is \{o, i, a, b, ab\}

The subsemigroup generated by \{ba, i, a, b\} \setminus \{ba\} is \{o, i, a, b, ba\}

For \{o, i, a, b\} \subseteq \{o\} is in the subsemigroup generated by \{o, i, a, b\} \setminus \{o\}

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For \{ba, i, a, b\} \subseteq \{ba\} is in the subsemigroup generated by \{ba, i, a, b\} \setminus \{ba\}

Therefore \(U\) is a maximal independent subset of \(S\). There may exist other Maximal Independent subsets in \(S\) but the number of elements in each must be the same. The rank of \(S\) is 3 - the number of elements in \(U\). Each maximal independent subset is called a basis of \(\langle U \rangle = \{o, i, a, b, ab, ba\} = S\).

2.0. **Definition** (Rank of Semigroup by Howie and Ribeiro (1999))

Given a finite semigroup \(S\).

1. \(r_1(S) = \max\{k : \text{every subset } U \text{ of cardinality } k \text{ in } S \text{ is independent}\}\).
2. \(r_2(S) = \min\{|U| : U \subseteq S, \langle U \rangle = S\}\).
3. \(r_3(S) = \max\{|U| : U \subseteq S, \langle U \rangle = S, U \text{ is independent}\}\).
4. \(r_4(S) = \min\{|U| : U \subseteq S, U \text{ is independent}\}\).
5. \(r_5(S) = \min\{k : \text{every subset } U \text{ of cardinality } k \text{ in } S \text{ generates } S\}\).

For a finite semigroup \(S\), it can be observed that

\[
r_1(S) \leq r_2(S) \leq r_3(S) \leq r_4(S) \leq r_5(S)
\]

Thus, \(r_1(S), r_2(S), r_3(S), r_4(S)\) and \(r_5(S)\) are, respectively, known as small rank, lower rank, intermediate rank, upper rank and large rank of \(S\). Here, the lower Rank is what is usually called the rank which has been extensively studied.

2.1. **Remark**

Our approach is to show first with construction of examples for infinite, commutative and non-commutative semigroups, where there exists finite basis and the number of independent elements is larger than the basis. A Basis is an independent subset and also a minimal generating set. The idea we intend to communicate here is that

i. As already known with semigroups, the minimal generating set is not always equal to the maximal independent subset. This prompted the modification of the definition of rank by Howie and Ribeiro in (5).

ii. Furthermore, the maximal independent subset does not always exist and thus fails to be a worthy tool for characterising ranks.

2.2. **Definition**

Let \(\{S, \ast\}\) be a Semigroup. Let \(H \subseteq S\) be a set. The closure of \(H\) in \(S\) is the Semigroup

\[
Cl(H) := \left\{ \prod_{j=1}^k a_j \left| \forall \{a_j\}_{j=1}^k \subseteq H, 1 \leq k \leq \infty \right. \right\}
\]

2.3. **Definition**

A subset \(G \subseteq S\) is a generator system in \(S\) if

\[
Cl(G) = S.
\]

2.4. **Definition**

A subset \(H \subseteq S\) is an independent system of semi-ring elements in \(S\) if \(\forall x \in H, x \notin \langle H \setminus \{x\} \rangle\).

2.5. **Definition**

A subset \(B \subseteq S\) is a basis in \(S\) if \(B\) is a minimal generating system in \(S\);
3.0. **Infinite Semigroup examples.**
In this section we will construct infinite semigroups generated by two or finite bases which
have infinite system of independent systems.

3.1. **Example for commutative semigroup.**
Let a commutative semigroup be generated by a basis \( B := \{ a, b \} \) of two independent elements.

Assume that both \( a, b \in B \) generate infinite cyclic Semigroups: \( C_a := \{ a^k \mid 1 \leq k < \infty \} \)
and \( C_b := \{ b^k \mid 1 \leq k < \infty \} \).

Let the semigroup \( S_c := \{ a^nb^m \mid 1 \leq n + m < \infty, n, m \in \mathbb{N} \} \) with the multiplication \((a^nb^m) \cdot (a^pb^q) := a^{n+p}b^{m+q} \). \( \forall 1 \leq (n + m)(p + q) \) and \( \{ n, m, p, q \} \subset \mathbb{N} \).

3.2. **Theorem**
\( H := \{ ab^j \mid 1 \leq j < \infty \} \subset S_c \) is an infinite independent system in \( S_c \).

**Proof.** Let \( ab^j \in H \) for \( j \in \mathbb{N}\setminus\{0\} \).

Then \( ab^j \in H\setminus\{ab^j\} \) and \( \forall x \in Cl(H\setminus\{ab^j\}) \ ab^j \neq x \in H \)
Otherwise \( x \notin H \Rightarrow x = a^pb^q, p \geq 2 \). Hence \( ab^j \notin Cl(H\setminus\{ab^j\}) \forall 1 \leq j < \infty \).

3.3. **Conclusion**
There exists a commutative semigroup generated with a two-element basis which has an infinite system of independent subset.

4.0. **Example of non-commutative semigroup**
Let a non-commutative semigroup be generated by a basis \( B := \{ a, b \} \) of two independent elements as a free semigroup of finite words created from \( B \):

\[ S_F := \{ a_1a_2 \ldots a_k \mid a_j \in B, 1 \leq j \leq k, 1 \leq k < 1 \} \] .

As in a free semigroup/group the operation is concatenation of words as elements of the
semigroup \( S_F \). Both \( a \) and \( b \) generates cyclic subsemigroups in \( S_F \).

\( S_{F,a} := \{ a^j, 1 \leq j < \infty \} \) and \( S_{F,b} := \{ b^j, 1 \leq j < \infty \} \).

4.1. **Theorem**
\( H := \{ ab^j \mid 1 \leq j < \infty \} \subset S_F \) is an infinite independent system in \( S_F \).

**Proof.**
Let \( ab^j \in H \) for \( a_j \in \mathbb{N}\setminus\{0\} \). Then \( ab^j \notin H\setminus\{ab^j\} \) and \( \forall x \in Cl(H\setminus\{ab^j\}) \Rightarrow ab^j \neq x \in H \). Otherwise \( x \notin H \Rightarrow x = \prod_{t=1}^{s} ab^q_t, s \geq 2, q_t \geq 1, 1 \leq t \leq s \).

Hence \( ab^j \notin Cl(H\setminus\{ab^j\}) \forall 1 \leq j < \infty \).

4.2. **Conclusion.**
There exists a non-commutative semigroup generated with a two-element basis which has an infinite system of independent subset.

**References**

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