ON COSETS AND INTERSECTION OF CYCLIC SUBSEMIGROUPS

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ABSTRACT: We give a brief discussion on cosets of cyclic subsemigroups S and partitioning of subsemigroups. Such partitioning provides a minimal generator of semigroups. We discuss intersection of cosets of cyclic subsemigroups, and formulate with a theorem, analog of a result on normal subgroups of a group for semigroups.

Key Words: Cosets, Semigroup, Subsemigroup, Cyclic Semigroup, partitioning, Commutative Semigroup.

1.0. Introduction and Preliminaries

In this paper, we develop specific tools for constructing generating system of semigroups. We explore the possibility of constructing generating sets by intersecting independent subsystems. A part is devoted to the studies of some behaviour of cyclic semigroups as we discuss about the problem of intersecting semigroups/groups etc. It can be shown that some selected family of connected cyclic semigroups can be embedded into a cyclic semigroup. It also discuss partly, the partition of semigroups using the cosets as congruence. (Lipcsey & Sampson, 2023) studied the structure of cyclic subsemigroups as graphs first with infinite cyclic subsemigroups as intersecting chains, where the intersection of any two infinite cyclic semigroups is an infinite cyclic subsemigroup of the two intersecting infinite cyclic semigroups. In their graph the nodes were semigroup elements, and the edges connected the \(a^k\) elements of the cycle to the consecutive \(a^{k+1}\) element. In this graph they did not consider multiplications between elements coming from two different chains. The graph structure is available in the original semigroup also and the smallest semigroup containing original cyclic semigroups can be mapped into the free semigroups. The whole idea was aimed at obtaining a perfect minimal candidate for generating the oversemigroup in which the generator itself is contained. Scholars have obtained similar results in alternative ways and one is by partitioning the oversemigroup into subsets that generate the whole, using congruences. Generating systems have been discussed in papers by (Lipcsey & Sampson, 2022). Some results here answer questions asked in (Lipcsey & Sampson, 2023, (Lipcsey & Sampson, 2019a) and (Lipcsey & Sampson, 2019b). Further explanations of terms used here can be found in (Howie, J., 1976).

1.1. Definition of Concepts

We will define some terms which our discussions in this paper are dependent on. We start with the simple structure of groupoid and then consider other concepts that are central to the work.

1.2. Definition (A Groupoid)

Let \(S\) be a set and \(\sigma : S \times S \rightarrow S\) be a binary operation that maps each ordered pair \((x, y)\) of \(S\) into an element \(\sigma(x, y)\) of \(S\). The pair \((S, \sigma)\) is called a groupoid.

1.3. Notation

The mapping \(\sigma\) is called the product of \((S, \sigma)\). We will write \((x \ast y)\) instead of \(\sigma(x, y)\). The element \(x \ast y \) (\(= \sigma(x, y)\)) is the product of \(x\) and \(y\) in \(S\).

1.4. Definition (Semigroup)

Let \(S\) be a set and \(* : S \times S \rightarrow S\) be a binary operation that maps each ordered pair \((x, y)\) of \(S\) into an element of \(S\). \(S\) is a Semigroup if \(\forall x, y, z \in S\)\

\((x \ast y) \ast z = x \ast (y \ast z)\)

1.5. Definition (Subsemigroup)

Let \((S, \ast)\) be an ordered pair, where \(S\) is a semigroup and "\(\ast\)" is the binary operation in \(S\). A subsemigroup of \((S, \ast)\) is a non-empty subset \(H\) of \(S\) which is closed under the multiplication of \(S\).
1.6. **Notation**
We will write $S$ instead of $\{S, \ast\}$ except where very necessary.

1.7. **Definition** (Commutative Semigroup)
Let $(S, \cdot)$ be a semigroup. $(S, \cdot)$ is called a Commutative Semigroup, if 
\[ \forall x, y \in S : x \cdot y = y \cdot x. \]

1.8. **Definition** (Monoid)
Let $(S, \ast)$ be an ordered pair where $\ast$ is an associative binary operation. $(S, \ast)$ is called a monoid, if there exists an element $e \in S$ such that $e \ast x = x \ast e = x \in S$. 

1.9. **Lemma.**
Let $\{S, \ast\}$ be a semigroup. If $S$ is not a monoid then there exists a set $M_S := S \cup \{e\}$ and an operation $\ast$ such that $e \ast e = e$; $e \ast a = a \ast e = a, \ \forall a \in S$ and $a \ast b = a \ast b, \ \forall a, b \in S$.

Then $\{M_S, \ast\}$ is a monoid.

1.10. **Remarks**
There is no essential difference between a monoid and a semigroup. A monoid $\{M, \ast\}$ with a neutral element $e \in M$ represents a semigroup if $M \setminus \{e\} = : S$, is a subsemigroup of $M$. In other words $S \subset M$ is a subsemigroup of $M$ if $S$ is an invariant subset of $M$ for the semigroup operation which means $S$ does not have any invertible element. Any semigroup is a subsemigroup of a monoid by the lemma 1.9.

1.11. **Lemma.**
Let $\{S, \ast\}$ be a semigroup. Let $H_\alpha \subset S$ be a subsemigroup $\forall \alpha \in \mathcal{A} \neq \emptyset$, then
\[ \emptyset \neq H = \bigcap_{\alpha \in \mathcal{A}} H_\alpha \]
is a subsemigroup of $S$.

**Proof.**
Let $u, v \in H$. Then $u, v \in H_\alpha, \ \forall \alpha \in A \implies u \ast v \in H_\alpha, \ \forall \alpha \in A \implies u \ast v \in H = \bigcap_{\alpha \in \mathcal{A}} H_\alpha$

2.0. **Intersections of subsemigroups.**
We now turn attention fully to the fundamental subsemigroup called cyclic semigroup. We begin with its definitions.

2.1. **Definition** (Cyclic Subsemigroup).
Let $\{S, \ast\}$ be a semigroup with a neutral element $e \in S$. Let $a \in S$. Then
\[ \{a^j \mid 0 \leq j \leq \infty, \ e = a^0\} =: \langle a \rangle \] is called a cyclic subsemigroup.

2.2. **Remark.**
The subset $\{a^j \mid j \in \mathbb{N}\} =: \langle a \rangle \subset S$ is a commutative subsemigroup (see 1.7 for definition of commutative semigroup): $\forall u, v \in \langle a \rangle \implies u = a^p, \ v = a^q \implies u \ast v = a^p \ast a^q = a^{p+q} = a^q \ast a^p = v \ast u \in \langle a \rangle$ by $p + q \in \mathbb{N}$.

2.3. **Remarks** (On Cosets of Cyclic Semigroups)
The definition of cosets given in section 2.4 below, serves as a very useful tool for the next theorem which of course provides the enablement for the partitioning of sets (as will be seen in later seen) especially where there is a single operation on the set. The definition is specialized for Cyclic Semigroups. We will see some properties of (cyclic) subsemigroups in the sections that follows.

Let $S$ be a semigroup and let $S_C$ be the set of its cyclic semigroups $S_C = \{(a) | a \in S\}$.

2.4. Definition (Cosets)

Let $c \in S_C$, $C := \langle c \rangle$ be a cyclic semigroup and let $b \in S$. Then

$bC := b \ast C = b(c) = \{b \ast c^k | k \in \mathbb{N}\}$ is a left coset formed by $b$ and $C$. Similarly

$Cb := C \ast b = (c)b = \{c^kb | k \in \mathbb{N}\}$ is a right coset formed by $b$ and $C$.

2.5. Theorem (An Analogue of Normal Subgroup).

Let $S$ be a semigroup with a neutral element and let $C \subseteq S$ be subsemigroups with neutral element. Then the following hold:

1. $C \ast C = \{a \ast b | a, b \in C\} = C$;
2. Let $C \subseteq H \subseteq S$ where $H, C$ are cyclic subsemigroups, $C = \langle t \rangle, t \in H$. Then the cosets fulfill $a \ast C = C \ast a$, $\forall a \in H$. In other words using terminology of group theory, $C$ is a normal subsemigroup of $H$.
3. Let $C = \langle t \rangle, t \in H$. Let $C_k := \{(t^k)s = \langle t^k \rangle s | 0 \leq s < \infty \} \subseteq C$. Then

$$(t^p \ast C) \cap (t^q \ast C) = \emptyset; \forall 0 \leq p, q < k, p \neq q.$$

Proof.

(i). Since $e \in C$, $C = e \ast C \subseteq \{a \ast b | a, b \in C\} = C \ast C \subseteq C \Rightarrow C \ast C = C$.

(ii). Since $H$ is a cyclic subsemigroup of $S$, it is a commutative semigroup. A subsemigroup $C \subseteq H$ is then a commutative subsemigroup of $H$ also. Then the left cosets are equal to the right ones: $a \ast C = C \ast a$, $\forall a \in H$ by the commutativity of $H$.

(iii). Let $C = \langle t \rangle = \{t^p | 0 \leq p < \infty\}, t \in H$. Let

$$C_k := \{(t^k)s = \langle t^k \rangle s | 0 \leq s < \infty \} \subseteq C.$$

The congruence relation of $\mod k$ equivalence is an equivalence relation. It partitions the natural numbers into $k$ pairwise disjoint equivalence classes

$$N_{k,q} := \{p | q \equiv p (\mod k)\}, 0 \leq q < k.$$

Accordingly, the cosets

$$\{t^p C_k = \{t^p(t^k)s \mid t^{p+k} \leq s < \infty\} = \{t^q \mid \alpha \in N_{k,q}\}, 0 \leq p < k.$$  

Hence let $t^p \ast C_k = \{t^q \mid \alpha \in N_{k,q}\}, t^q \ast C_k = \{t^q \mid \alpha \in N_{k,q}\}, \forall 0 \leq p, q < k, p \neq q$. Then

$$N_{k,p} \cap N_{k,q} = \emptyset \Rightarrow (t^p \ast C_k) \cap (t^q \ast C_k) = \emptyset; \forall 0 \leq p, q < k, p \neq q.$$

3.0. Further Illustrations

We present some lemmas to give more light to the theorem of section 2.5.

3.1. Lemma [7].

Let $\langle S, \ast \rangle$ be a semigroup with a neutral element, let $s \in S \setminus \{e\}, \langle s \rangle := C \subseteq S$ be a cyclic subsemigroup. Let $n, m, k \in \mathbb{N}$, let $n|m$ and $k < n$. Then the cosets $C_{n,k}$ and $C_{m,k}$ fulfill

$$C_{n,k} \supset C_{m,k}.$$  

Proof.
\[
C_{n,k} = \{ s^{k+nt} \mid 0 \leq t < \infty \} \text{ and } C_{m,k} = \{ s^{k+mt} \mid 0 \leq t < \infty \}.
\]

Then \( m = na \) for an \( a \in \mathbb{N} \) by \( n|m \).

Therefore
\[
N_{m,k} = \{ v \mid v = k + mt = k + nat, \ t \in \mathbb{N} \} \subset \{ k + nt \mid t \in \mathbb{N} \} = N_{n,k}.
\]

Hence
\[
C_{m,k} = \{ s^v \mid v \in N_{m,k} \} \subset \{ s^w \mid w \in N_{n,k} \} = C_{n,k}.
\]

3.2. **Lemma.**

Let \( \{ S, * \} \) be a semigroup with a neutral element, let \( S \in S \setminus \{ e \} \), \( (s) = : C \subset S \) be a cyclic subsemigroup.

Let \( n, m \in \mathbb{N} \), let \( n|m \) and \( k_1, k_2 \in \mathbb{N}, k_1 \neq k_2, \ 0 \leq k_1, k_2 < n \).

Then
\[
C_{n,k_1} \cap C_{m,k_2} = \emptyset.
\]

**Proof.** By the lemma of 2.6., \( C_{n,k_1} \supseteq C_{m,k_2} \) holds

By statement (iii) of the theorem of 2.5.,
\[
C_{n,k_1} \cap C_{n,k_2} = \emptyset. \tag{3}
\]

Intersecting both sides of equation (2) by \( C_{n,k_2} \) and applying lemma 2.6 leads to
\[
\emptyset = C_{n,k_1} \cap C_{n,k_2} \supseteq C_{n,k_1} \cap C_{m,k_2} \tag{4}
\]

3.0. **Infinite Systems of Cosets.**

In the previous section we defined cosets of cyclic subsemigroups in definition 2.3. Now we will define cosets of subsemigroups with special attention to subsemigroups of commutative (sub) semigroups.

3.1. **Definition (Left Coset and Right Coset)**

Let \( \{ S, * \} \) be a semigroup and let \( K \subset S \) be a subsemigroup.

Let \( a \in S \setminus K \) then \( a * K \subset S \) is a left coset and \( K * a \subset S \) is a right coset of \( K \).

3.2. **Lemma (Left and Right Cosets: Commutative Semgroup)**

Let \( C \subset S \) be a commutative subsemigroup and let \( K \subset C \subset S \) be a subsemigroup of \( C \). Then a left coset of \( K \), \( a * K \subset S \) and a right coset of \( K \), \( K * a \subset S \) are equal:
\[
a * K = K * a, \ \forall a \in C \setminus K.
\]

**Proof.** By commutativity of \( S \), the products of elements of \( K \) by \( a \in C \setminus K \), which can be written as \( a * K \) or \( K * a \), \( \forall a \in C \setminus K \), would yield exactly the same value. So \( a * K = K * a, \forall a \in C \setminus K \).

3.3. **Conclusion and Recommendation.**

The commutative subsemigroup \( C \) in the lemma 3.2 will be mostly cyclic subsemigroup.

In the previous sections we considered semigroups with finite system of pairwise disjoint cosets. It will be interesting to see how this can be handled for the case of infinite system of cosets in a cyclic subsemigroup.

It will also be interesting to see how using the concept of ideal will turnout for such structure. If \( C \subset S \) is a subsemigroup of the semigroup \( S \), the product of the elements \( a \in C \setminus K \), where
\[
K \subset C, \text{ with any other element of } S \text{ that generates the left cosets } a * K \text{ and right cosets } K * a \text{ which lies on any part of the subsemigroup } S. \text{ How do we handle such multiplication if we insist that the product lies in } C? \text{ This leads us the concept of Ideals}
in semigroup, preferred by many semigroup theorist for obtaining the green’s congruences, the $H$-congruence being an equivalent relation is popularly known to partition a semigroup like the coset does in groups.

References


