A typical semigroup in the class of generalized $*$-bisimple ample ω-semigroups

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ABSTRACT: A class of $*$-bisimple ample ω-semigroups whose certain $\mathcal{H}^*$-classes contain no regular elements was considered in [3], this class of $*$-bisimple ample ω-Semigroups are seen to be generalized. The characterization of this class of semigroups rather involves the use of bisystems with certain structure mappings and a binary multiplication together with some closure axioms. The characterization of this class of semigroups were given in [4]. This paper is a follow up of [3] and [4] which presents a generalized theorem for a semigroup in this class. Moreover, a typical example of a semigroup in this class of $*$-bisimple ample ω-semigroups is constructed to offer an insight into this class of semigroups.

Keywords: ***Cancellative monoids, abundant semigroups, adequate semigroups, ample semigroups, ω-chain, *-bisimple semigroups, binary array of bisystems of cancellative monoids.***

Throughout this paper S denotes a semigroup (except stated otherwise), terminologies are curled from [2] and [3].

I Preliminaries

The following are well known results.

Definition 1.1

Let $a, b \in S$, then Green's $*$-relations are defined as follows:

$\mathcal{L}^* = \{(a,b) \in S \times S: \forall\, x, y \in S^1\ ax = ay \iff bx = by\}$

$\mathcal{R}^* = \{(a,b) \in S \times S: \forall\, x, y \in S^1\ xa = ya \iff xb = yb\}$

$\mathcal{H}^* = \{(a,b) \in S \times S: (a,b) \in \mathcal{L}^* \cap \mathcal{R}^*\}$

$\mathcal{D}^* = \{\exists\, c \in S: (a,c) \in \mathcal{L}^*: (c,b) \in \mathcal{R}^* \text{ or } a\mathcal{L}^*c\mathcal{R}^*b\} = \mathcal{L}^* \vee \mathcal{R}^*$

Definition 1.6

Let S be a semigroup, and let $a, b \in S$, the relation $\widetilde{\mathcal{D}}$ on S is defined by:

$a\widetilde{\mathcal{D}}b \iff a^*\mathcal{D}b^*, a^+\mathcal{D}b^+$.

Lemma 1.7 [4]

On S, the relation $\widetilde{\mathcal{D}}$ satisfies the inclusion $\mathcal{D}(S) \subseteq \widetilde{\mathcal{D}}(S) \subseteq \mathcal{D}^*(S)$.

Proof

Observe that if $(a^*, b^*) \in \widetilde{\mathcal{D}}(S), (a^+, b^+) \in \widetilde{\mathcal{D}}(S) \Rightarrow (a,b) \in \mathcal{D}^*(S) \Rightarrow \widetilde{\mathcal{D}}(S) \subseteq \mathcal{D}^*(S)$.

We now show that $\mathcal{D}(S) \subseteq \widetilde{\mathcal{D}}(S)$.

Observe that if $(a,b) \in \mathcal{D}(S)$, then there exists $c \in S$, such that $a\mathcal{R}c\mathcal{L}b$ with $a^* \in \mathcal{L}_a^*$ and $b^* \in \mathcal{R}_a^*$ and so $(c, a^*) \in \mathcal{L}$. Now because $(c,b) \in \mathcal{R}$ then there exists $x, y \in S^1$ such that $cx = b, by = c$ thus $cxy = c$ and since $c^* = a^*$ it follows that $a^*xy = a^*$ so that $(a^*x, a^*) \in \mathcal{R}$ showing the regularity of a^*x. But $(c, a^*) \in \mathcal{L}^*$ and $(b, b^*) \in \mathcal{L}^*$ and so, for any $s, t \in S^1$, $a^*xs = a^*xt \iff cxs = cxt$ and $bs = bt \iff b^*s = b^*t$ and then $(a^*x, b^*) \in \mathcal{L}$ and therefore $a^*\mathcal{R}a^*x\mathcal{L}b^* \Rightarrow (a^*, b^*) \in \mathcal{D}(S)$. Similarly, $(a^+, b^+) \in \widetilde{\mathcal{D}}(S)$. □

Lemma 1.8 [4]

On S, the following are equivalent;

i. $\widetilde{\mathcal{D}}(S) = \mathcal{D}^*(S)$

ii. Every non-empty $\mathcal{H}^*$-class of S contains some regular element.

Proof

Suppose i holds, then from lemmas 1.3 and 2.1 above $(a^*, b^*) \in \widetilde{\mathcal{D}}(S) \Rightarrow (a^*, a^*x) \in \mathcal{R}, (a^*x, b^*) \in \mathcal{L}$. For simplicity let $e = a^*, f = b^*$ and $v = a^*x$. Thus, by lemma 1.4, $(e,v) \in \mathcal{R} \Rightarrow ev = v$. Similarly, $(v,f) \in \mathcal{L} \Rightarrow vf = v$. Again, $(e,v) \in \mathcal{R} \Rightarrow \exists\, s, t \in S^1$ such that $vs = e$ and $te = v$. Now let $v' = fse$, so that $vv'v = v(fse)v = (vf)s(ev) = vsv = ev = v$ and so $v' \in V(v)$, where,

$V(v) = \{v \in S: vv'v = v \text{ and } v'vv' = v'\}$. Also, $vv' = v(fse) = (vf)se = vse = ee = e$. But also, since $(v,f) \in \mathcal{L}$ then there exists $t \in S^1$ such that $tv = f$, thus, $v'v = (fse)v = fsev = fsv = tvsv = tev = tv = f$. It now follows easily that $v' \in \mathcal{L}_e \cap \mathcal{R}_f = \mathcal{L}_u \cap \mathcal{R}_u = \mathcal{H}_u$ for some $u \in \mathcal{H}_{v'}$.

But $\mathcal{H} \subseteq \mathcal{H}^*$.

ii ⇒ i is straight forward since there exists an inverse v' of v in $\mathcal{L}_e \cap \mathcal{R}_f$ such that $e = vv', f = v'v$ and so $(e,v) \in \mathcal{R}$ and $(v,f) \in \mathcal{L} \Rightarrow (e,f) \in \mathcal{D} \Rightarrow (a^*, b^*) \in \mathcal{D} \subseteq \widetilde{\mathcal{D}}(S) = \mathcal{D}^*(S)$ □

The following are results from [3]

Consider the set of triples by:

$S(\boldsymbol{B}, d, \theta, \varphi) = \{(m, x, n): m, n \in N, x \in B_{\bar{m},\bar{n}}\}$, for simplicity, let $S(\boldsymbol{B}, d, \theta, \varphi) = S$
Suppose that $(m, x, n), (p, y, q) \in S(\boldsymbol{B}, d, \theta, \varphi)$, define the multiplication on $S(\boldsymbol{B}, d, \theta, \varphi)$ by:

$$(m, x, n)(p, y, q) = (m - n + t, x\theta^{t'-n_d} y \varphi^{t'-p_d}, q - p + t) \quad [1.1]$$

where $t = \max(n, p)$, $t' = \max(n_d, p_d)$.

Theorem 1.9 [3]
$S = S(\boldsymbol{B}, d, \theta, \varphi)$ as defined in [1.1] is a semigroup. [(see 3]

Lemma 1.10 [3]
Let S be as defined in [1.1], then idempotents of S are of the form $(m, e_m, m), m, n \in N, e_m \in B_{\bar{m},\bar{n}}$.

Lemma 1.11 [3]
Let S be as defined in [1.1] and let $a = (m, x, n) \in S$, and let $x \in M_m$, where x is a unit, then the inverse of a, is of the form $a^{-1} = (n, y, m)$, where $y = x^{-1}$ and $m = n(\mathrm{mod} d)$.

Lemma 1.12 [3]
Let $a = (m, x, n), f_n = (n, e_{\bar{n}}, n), f_m = (m, e_{\bar{m}}, m) \in S$. Then for all $u = (h, y, k), v = (f, z, g) \in S$, then:

i. $a\mathcal{L}^* f_n$ and
ii. $a\mathcal{R}^* f_m$

Lemma 1.13[4]
Let $a = (m, x, n), b = (p, y, q)$

i. $a\mathcal{L}^* b \Leftrightarrow n = q$ and dually
ii. $a\mathcal{R}^* b \Leftrightarrow m = p$

Following lemmas 1.12 and 1.13 we have

Lemma 1.14[4]
Let S be as defined in [1.1], then *S is abundant.*

Lemma 1.15[4]
S as defined in [1.1] *is adequate*

Theorem 1.16[4]
$S = S(\boldsymbol{B}, d, \theta, \varphi)$ is an ample ω - semigroup.

Theorem 1.17[4]
S is a $*$- bisimple ample ω-semigroup

Lemma 1.18 [2]
S is $*$- *simple if and only if* $\mathcal{R}_0^* = \mathcal{H}_{0,0}^*$

Theorem 1.19[2]
Let *S be an ample ω-semigroup in which $\mathcal{D}^* = \widetilde{\mathcal{D}}$. Then the set $\mathcal{Q}_s$ of regular elements in S is an inverse ω-semigroup with ω-chain of idempotents $E(S) \subseteq \mathcal{Q}_s$. Moreover, $\mathcal{Q}_s$ is simple or bisimple according as S is $*$- simple or $*$- bisimple.*

Lemma 1.20[2]
The following conditions are equivalent in S.

i) $(e_m, e_n) \in \mathcal{D}^*$
ii) $\mathcal{H}_{m,n} \neq \emptyset$
iii) $\mathcal{H}_{m,n}\ m \equiv n(\mathrm{mod} d)$

Lemma 1.20[2]
Every element of S is uniquely expressible in form:
$a^{-k} x_i a^h, k, h \in N, 0 \leq i \leq d - 1, x_i \in \mathcal{H}_{i,i}^*$.

Following [1] and [2], we have the following;
Let $\mathcal{R}_m^*(\mathcal{L}_n^*)$ be the $\mathcal{R}^*(\mathcal{L}^*)$ classes containing the idempotents $e_m(e_n)$, define;
$\mathcal{L}_n^* = \{a \in S: a\mathcal{L}^* e_n\}$
$\mathcal{R}_m^* = \{a \in S: e_m \mathcal{R}^* a\}$.
We then see that;
$\mathcal{L}_n = \mathcal{L}_n^* \cap \mathcal{Q}_s$
$\mathcal{R}_m = \mathcal{R}_m^* \cap \mathcal{Q}_s$, thus, we have that
$\mathcal{H}_{m,n} = \mathcal{H}_{m,n}^* \cap \mathcal{Q}_s$.
Where $\mathcal{L}_n$, $\mathcal{R}_m$ and $\mathcal{H}_{m,n}$ are the corresponding classes in Green's equivalences. It follows that
$\mathcal{D}(\mathcal{Q}_s) \subseteq \mathcal{D}^* S) \cap (\mathcal{Q}_s \times \mathcal{Q}_s)$.

If S is a $*$- bisimple ample ω-semigroup such that $\mathcal{D}^* = \widetilde{\mathcal{D}}$, then lemma 1.8 above, shows that each $\mathcal{H}^*$-class contains regular elements. Consequently;
$\mathcal{H}_{m,n} \neq \emptyset$ for all $m, n \in N$ and $e_n, e_m \in E(S)$.

Clealy $\mathcal{H}_{m,n}^* = \{a \in S: a\mathcal{L}^* e_n, a\mathcal{R}^* e_m\} = \mathcal{L}_n^* \cap \mathcal{R}_m^*$
Now observe from these that:
$e_m \mathcal{R}^* a \mathcal{L}^* e_n$ and so $(e_m, e_n) \in D^*$
But $(e_m, e_n) \in D^* \Longrightarrow (e_m, e_n) \in \widetilde{D}$

Thus $\mathcal{H}^*_{m,n} \neq \emptyset$, then clearly $\mathcal{H}^*_{m,n}$ is an $\mathcal{H}^* -$ class of S.

Theorem 1.19[2]
Let T be a cancellative monoid and θ an endomorphism of T. Then the generalized Bruck- Reilly extension $BR^*(T,\theta)$ is a $*$-bisimple ample ω-semigroup with $\mathcal{D}^* = \widetilde{\mathcal{D}}$. Conversely, every $*$- bisimple ample ω-semigroup with $\mathcal{D}^* = \widetilde{\mathcal{D}}$ is isomorphic to one of this type.

2.0 Semigroups in the class of generalized $*$- bisimple inverse ω-semigroups.

In a $*$- bisimple inverse ω-semigroup in which $\mathcal{D}^*(S) \neq \widetilde{\mathcal{D}}(S)$, its regular subsemigroup $\mathcal{Q}_s$ is not a $*$- bisimple inverse ω-semigroup, since by lemma 1.18 there exists an $\mathcal{H}^*$-class of S with no regular elements. If $\mathcal{H}^*_{m,n}$ is such $\mathcal{H}^*$-class, then $\mathcal{H}_{m,n} = \mathcal{H}^*_{m,n} \cap \mathcal{Q}_s = \emptyset$, consequently, $(e_m, e_n) \notin \mathcal{D}(\mathcal{Q}_s)$.
But Munn [9] has shown that in a simple inverse ω-semigroup, $\mathcal{H}_{m,n} \neq \emptyset$ if and only if $m \equiv n(\mathrm{mod} d)$ and consequently, $\mathcal{H}^*_{m,n} \cap \mathcal{Q}_s \neq \emptyset$ if and only if $m \equiv n(\mathrm{mod} d)$. That is $\mathcal{H}^*_{m,n} \cap \mathcal{Q}_s \neq \emptyset$ if and only if $\mathcal{Q}_s$ is a simple inverse ω-semigroup.
Thus, by implication from lemma 2.6 [9] we see that $\mathcal{Q}_s$ is;

a. a chain of groups, or
b. a simple inverse ω-semigroup with d- $\mathcal{D}$-classes, or
c. an extension of a simple inverse ω-semigroup by a proper kernel.

We now examine the different classes of $*$- bisimple ample ω-semigroup in which the regular subsemigroup $\mathcal{Q}_s$ corresponds to each of the three cases above.
Let S be $*$- bisimple ample ω-semigroup in which $\mathcal{Q}_s$ is not bisimple.

a) S with $\mathcal{Q}_s$ a chain of groups is briefly described in the following lemma.

Lemma 2.1
Let S be $*$- bisimple ample ω-semigroup such that $\mathcal{H}^*_{m,n} \cap \mathcal{Q}_s \neq \emptyset$ if and only if $m = n$. Then $\mathcal{Q}_s$ is an ω- chain of groups.
Proof:
If $\mathcal{Q}_s$ is not bisimple, and $\mathcal{H}^*_{m,n}$ contain regular elements if and only if $m = n$, then $\mathcal{H}_{m,n} = \mathcal{H}^*_{m,n} \cap \mathcal{Q}_s = \emptyset$ for all natural numbers $m, n, m \neq n$.
The subsemigroup
$T = \cup \{\mathcal{H}^*_{m,n} : m \in N\}$ is an ω- chain of cancellative monoids and a full subsemigroup of S by theorem 2.8 [2]. Moreover, $\mathcal{Q}_s \cap T = \cup \{\mathcal{H}^*_{m,n} : m, n \in N\} = \mathcal{Q}_s$ is an ω- chain of groups consisting groups of units of $\mathcal{H}^*_{m,n}, m, n \in N$. We observe that, if $a \in \mathcal{H}^*_{m,n}, b \in \mathcal{H}^*_{p,q}$, where $p \neq q$, then $\mathcal{H}^*_{p,m} = \emptyset$ so that with $a = e_m$ and $a^* = e_p$, then $(e_m, e_n) \notin \mathcal{D}(\mathcal{Q}_s)$, hence $(a, b) \notin \mathcal{D}(S)$. Therefore $\mathcal{D}^*(S) \neq \widetilde{\mathcal{D}}(S)$.

b) If $\mathcal{H}^*_{m,n}$ contains a regular element for $m \neq n$ and e_n. $\mathcal{H}^*_{0,n-m} = \mathcal{H}^*_{m,n} (m < n)$, we then see that $\mathcal{H}^*_{0,n-m}$ also contains a regular element.

Lemma 2.2
Let S be $*$- bisimple ample ω-semigroup with $\mathcal{D}^* \neq \widetilde{\mathcal{D}}$. If $\mathcal{H}^*_{0,n} \cap \mathcal{Q}_s \neq \emptyset$ for a non-zero number n then $\mathcal{Q}_s$ is a simple inverse ω-semigroup with d $\mathcal{D}$-classes.
Proof:
If $\mathcal{H}^*_{0,n} \cap \mathcal{Q}_s \neq \emptyset$. Consequently, $\mathcal{R}_0 = \mathcal{R}^*_0 \cap \mathcal{Q}_s \neq \emptyset$ so that $\mathcal{Q}_s$ must be a simple inverse ω-semigroup by [9].
Now if d is the least positive integer such that $\mathcal{H}_{0,d}$ contains regular elements, then $\mathcal{H}_{0,d} \neq \emptyset$ and for all $0 < i, j < d-1, i \neq j$, the $\mathcal{H}^*$-classes $\mathcal{H}_{i,0}, \mathcal{H}_{0,i}$ and $\mathcal{H}_{i,j}$ contain no regular elements, consequently, $(e_i, e_j) \notin \mathcal{D}$ for $i \neq j$. Therefore, each $e_i, 0 \leq i \leq d-1$ belong to a distinct $\mathcal{D}$-class and so from [9], $\mathcal{Q}_s$ is simple with d $\mathcal{D}$-classes and the prove of the lemma is complete.
However, it is observed that if $\mathcal{H}^*_{0,d} \cap \mathcal{Q}_s \neq \emptyset$, then $\mathcal{H}^*_{0,kd} \cap \mathcal{Q}_s \neq \emptyset$ and $\mathcal{H}^*_{kd,0} \cap \mathcal{Q}_s \neq \emptyset$ so that with $0 < i < d-1$, $\mathcal{H}^*_{0,kd+i}.\mathcal{H}^*_{kd,0} \subseteq \mathcal{H}^*_{0,i} = \emptyset$ and consequently, $\mathcal{H}^*_{0,kd+i} \cap \mathcal{Q}_s \neq \emptyset$. This together with Lemma 1.20 above shows that $\mathcal{H}^*_{m,n}$ contains regular element if and only if $m = n(\mathrm{mod} d)$.
Conversely, suppose S be $*$- bisimple ample ω-semigroup such that $\mathcal{Q}_s$ is a simple inverse ω-semigroup with d $\mathcal{D}$-classes. Then from lemma 1.20, $\mathcal{H}^*_{m,n} \cap \mathcal{Q}_s \neq \emptyset$ if and only if $m = n(\mathrm{mod} d)$ and if and only if $(e_i, e_j) \in \mathcal{D}(\mathcal{Q}_s)$. However, to see that $\mathcal{D}^*(S) \neq \widetilde{\mathcal{D}}(S)$, let $a \in \mathcal{H}^*_{0,i}, b \in \mathcal{H}^*_{0,j}, i \neq j, 0 \leq i, j \leq d$. But
$a^+ = e_i, b = e_j$ and $(e_i, e_j) \notin \widetilde{\mathcal{D}}$, consequently, $(a, b) \notin \widetilde{\mathcal{D}}$. Thus, we have proved

Lemma 2.3
$\mathcal{H}^*_{m,n}$ contains regular elements if and only if $m = n(\mathrm{mod} d)$

c) S with $\mathcal{Q}_s$ as an extension of a simple inverse ω-semigroup.

Lemma 2.4
Let S be $*$- bisimple ample ω-semigroup with $\mathcal{D}^*(S) \neq \widetilde{\mathcal{D}}(S)$. If $\mathcal{H}^*_{m,n} \cap \mathcal{Q}_s \neq \emptyset$ and $\mathcal{Q}_s$ is not bisimple for $m \neq n$, then $\mathcal{Q}_s$ is an extension of a simple inverse ω-semigroup.
Proof:
Suppose that $\mathcal{H}^*_{m,n} \cap \mathcal{Q}_s \neq \emptyset$ for $m \neq n$ then $\mathcal{Q}_s$ is not an ω-chain of groups. Since $\mathcal{Q}_s$ is not a simple inverse ω-semigroup, the facts in Lemma 1.7 shows that $\mathcal{R}_0 = \mathcal{R}^*_0 \cap \mathcal{Q}_s = \mathcal{H}_{0,0}$. If i is the least positive integer such that $\mathcal{R}_i = \mathcal{R}^*_i \cap \mathcal{Q}_s \neq \mathcal{H}_{i,i}$.
Let $S_i = e_i S e_i$, then $e_i \mathcal{H}^*_{m,n} e_i = \mathcal{H}^*_{m,n} \in S_i$ and $\mathcal{Q}_{S_i}$ is a simple inverse ω-semigroup in view of [9] and Lemma 1.20.
Let $a = (m, x, n) \in \mathcal{H}^*_{m,n}$, then recall from Lemma 1.11 that a is regular if and only if $x \in M_m$ with an inverse x^{-1}, that is $\bar{m} = \bar{n}$. Thus, it is evident that $\mathcal{H}^*_{m,n}$ contains regular element if and only if $m \equiv n(\mathrm{mod} d)$. Thus, if $a = (m, x, n) \in \mathcal{H}^*_{m,n}$ is regular, then $(f_m, a) \in \mathcal{R}$ and $(a, f_n) \in \mathcal{L}$, so that

$f_m \mathcal{R}a\mathcal{L} f_n$ and so $(f_m, f_n) \in \mathcal{D}$. Conversely, if $(f_m, f_n) \in \mathcal{D}$, then $a = (m, e_{\bar{n}}, n)$ is a regular element such that $f_m \mathcal{R}a\mathcal{L} f_n$ then $a \in \mathcal{H}_{m,n}$ and evidently $\mathcal{H}_{m,n} \neq \emptyset$. Hence, the precise set of regular elements are given by:
$\mathcal{H}_{m,n} = \{(m, x, n): x.x^{-1} \in M_m, m \equiv n \equiv \bar{m}(\mathrm{mod} d)\}$. However, with $\mathcal{H}_{m,n} \neq \emptyset$, evidently,
$(f_m, f_n) \in \mathcal{D}$ and we have proved

Lemma 2.5

Let $f_m = (m, e_{\bar{m}}, m), f_n = (n, e_{\bar{n}}, n) \in S$, then $(f_m, f_n) \in \mathcal{D}$ if and only if $\mathcal{H}^*_{m,n}$ contains a regular element. □

Lemma 2.6

Let $u = (m, x, n), v = (p, y, q) \in S$, then $u\widetilde{\mathcal{D}}v$ if and only if $m \equiv p(\mathrm{mod} d)$ and $n \equiv q(\mathrm{mod} d)$.

Proof

Recall that by the definition of $\widetilde{\mathcal{D}}$, $u\widetilde{\mathcal{D}}v \Rightarrow u^* = f_m \mathcal{D}^* f_n = v^*$ and $u^+ = f_m \mathcal{D}^* f_n = v^+$. But by Lemma 2.5, $(f_m, f_n) \in \mathcal{D}$ if and only if $m \equiv n(\mathrm{mod} d)$ and then the prove is complete.

Thus, lemma 2.6 shows that $\widetilde{\mathcal{D}} \neq \mathcal{D}^*$.

We prove below that a regular subsemigroup $\mathcal{Q}_s$ of S is a simple inverse ω-semigroup with d D-classes.

Lemma 2.7

Let $\mathcal{Q}_s$ be the set of regular elements of S, then $\mathcal{Q}_s$ is a simple inverse semigroup with d-$\mathcal{D}$-classes.

Proof:

Let $u = (m, x, n) \in \mathcal{Q}_s$. Then $m \equiv n(\mathrm{mod})d$. $f_m \mathcal{D} f_n$ and $u^{-1} = (n, x^{-1}, m)$ where x^{-1} is the inverse of x in T_m. Therefore $\mathcal{Q}_s = \{(m, x, n)\} |\ m \equiv n(\mathrm{mod}\ id), x \in g(T_i)$
$i = 0,1,2 \ldots \ldots d-1, m, n, \in N$ where $g(T_i)$ is the group of units of $T_i, 0 \leq i \leq d-1$.

Let I be an ideal of $\mathcal{Q}_s$, if $u \in I$, then evidently, $f_m = u.u^{-1} \in I$. so that I contains an idempotent f_m and consequently for all $k \in N$, $f_k.f_m \in I$. If in particular we let $k > m$, where $k \equiv 0(\mathrm{mod}\ d)$. Then $f_k = f_k.f_m \in I$. But $f_k \mathcal{D} f_0$ hence $f_0 \in I$. Thus, for all $u \in \mathcal{Q}_s$,
$u = u.f_0 \in I$ consequently, $\mathcal{Q}_s \subseteq I$. Therefore, $I = \mathcal{Q}_s$. Since $E(S) \subseteq \mathcal{Q}_s$. $\mathcal{Q}_s$ is a simple inverse ω-semigroup with non-trivial ideals. Hence, $\mathcal{Q}_s$ has d-$\mathcal{D}$-classes by Lemma 2.5 and the prove is complete. ⊡

Combining all the facts from [2], [3], [4] together with lemmas 2.4, 2.5, 2.6 and 2.7, we obtain the following result;

Theorem 2.8:

Let $S = S(M, d, \theta, \varphi)$ be a set of triples on a binary array of bisystems $(M, d, 0, \varphi)$ subject to the operation as in [1.1]. Then S is a $*$-bisimple ample ω-semigroup in which $\mathcal{D} \neq \mathcal{D}^*$ and the set of regular elements in S is a simple inverse ω-semigroup with d-$\mathcal{D}$-classes ⊡

Now we construct a typical example generalized $* -$*bisimple* ample ω $-$semigroup

3.0 $* -$*bisimple* ample ω $-$semigroup with $\mathcal{D}^* \neq \widetilde{\mathcal{D}}$

Let T be a cancellative monoid and I an ideal of T. Let
$S = \{(m, x, n): x \in T$ if $m = n(\mathrm{mod} 2)\} \cup \{(m, x, n), x \in I,$ if $m \neq n(\mathrm{mod} 2)\}$ for
$m, n \in N$. Define multiplication on S by:
$(m, x, n)(p, y, q) = (m - n + t, xy, q - p + t)$ where $t = \max(n, p)$. [3.1]

Theorem 3.1

S is a semigroup with respect to the operation defined in [3.1] above.

Proof:

By considering the arbitrary elements $(m, x, n), (p, y, q) \in S$, then their product
$(m, x, n)(p, y, q) = (m - n + t, xy, q - p + t)$ where $t = \max(n, p)$ and
$(m - n + t) \equiv (q - p + t)\mathrm{mod} 2$ for $xy \in T$ or $(m - n + t) \not\equiv (q - p + t)\mathrm{mod} 2$ for $xy \in I$
Thus, this product is closed in S.

For $(m, x, n), (p, y, q)$ and $(r, z, s) \in S$, then
$[(m, x, n)(p, y, q)](r, z, s) = [m - n + t, xy, q - p + t](r, z, s)$, where $t = \max(n, p)$
$= [m + p - n - q + t_1, (xy)z, s - r + t_1]$, where $t_1 = \max[q - p + \max(n, p), r]$
$= [m - n + p - q + t_1, (xy)z, s - r + t_1]$
$= [m - n + t_3, (xy)z, s - r + q - p + t_3]$ [3.2]
Hence $(m - n + t_3) \equiv (s - r + q - p + t_3)\mathrm{mod} 2$ for $(xy)z \in T$ or
$(m + p - n - q + t_1) \not\equiv (s - r + t_1)\mathrm{mod}\ 2$ for $(xy)z \in I$.
Also:
$(m, x, n)[(p, y, q)(r, z, s)] = (m, x, n)[p - q + t_2, yz, s - r + t_2], t_2 = \max(q, r)$
$= [m - n + t_3, x(yz), s - r + q - p + t_3]$, where $t_3 = \max[n, p - q + \max(q, r)]$ [3.3]
$(m - n + t_3) \equiv (s - r + q - p + t_3)\mathrm{mod} 2$ for $x(yz) \in T$ or

$(m-n+t_3) \not\equiv (s-r+q-p+t_3) \text{mod} 2$ for $x(yz) \in I$
By comparing [3.2] and [3.3], noting that the outer coordinates are bicyclic and are associative while T is a monoid and I an ideal of T, so that $(xy)z = x(yz)$. Thus
$[(m,x,n)(p,y,q)](r,z,s) = (m,x,n)[(p,y,q)(r,z,s)]$ and then S is a semigroup.

$(m,x,n) \in S$, suppose that $(m,x,n)(m,x,n) = (m,x,n)$ then
$(m-n+t, x^2, n-m+t) = (m,x,n), t = \max(m,n)$.
$m-n+t = m \Longrightarrow n = t$ [3.4]
$n-m+t = n \Longrightarrow m = t$ [3.5]
By [3.4] and [3.5] we have $m = n$
Now $x^2 = x$ if and only if $x = 1 \in T$
We have proved that

Lemma 3.2
The idempotents of S are of the form $(m,1,m), m \in N$. □

Definition 3.3
Let $E_S = \{(m,1,m): m \in N\}$ be the set of idempotents of S

Lemma 3.4
E_S is ω – chain of idempotent.

Proof:
Already from above, the idempotents of S are of the form $E_S = \{(m,1,m): m \in N\}$
But if $(m,1,m) \geq (n,1,n)$, for $n \geq m$, then E_S forms an ω −chain.

Let $(m,x,n),(p,y,q) \in S$, and let $(i,a,j),(h,b,k) \in S^1$ such that
$(m,x,n)(i,a,j) = (m,x,n)(h,b,k)$ If and only if $(p,y,q)(i,a,j) = (p,y,q)(h,b,k)$
That is $(m-n+t, xa, j-i+t) = (m-n+u, xb, k-h+u)$ [3.6]
Where $t = \max(n,i), u = \max(n,h)$
And
$(p-q+v, ya, j-i+v) = (p-q+w, yb, k-h+w)$. [3.7]
where $v = \max(q,i), w = \max(q,h)$
By comparing [3.6] and [3.7], we have $i = h, j = k$
But by the cancellation law in T, $xa = xb \Longrightarrow a = b$ and so $(m,x,n)\mathcal{L}^*(\mathcal{R}^*)(h,1,h)$ and we have proved;

Lemma 3.5
$(m,x,n)\mathcal{L}^*(h,1,h)$
$(m,x,n)\mathcal{R}^*(k,1,k)$ □

Lemma 3.6
S is adequate.
Proof:
Since the idempotents of S are of the form $(m,1,m)$ and $(n,1,n)$ in E_S then clearly
$(m,1,m)(n,1,n) = (m-m+t, 1, n-n+t)$
where $t = \max(m,n)$.
$(m,1,m)(n,1,n) = (m-m+t,1,n-n+t) = (t,1,t) = (n,1,n)(m,1,m)$ showing that $\forall\, e,f \in E_S, ef = fe$. Which shows that S is adequate.

Lemma 3.7
S is an ample semigroup
Proof:
But since S is adequate, then let $e = (p,1,p)$ and let $a = (m,x,n)$, then
$ea = (p,1,p)(m,x,n) = (t,x,n-m+t), t = \max(p,m)$ [3.8]
Hence $ea = (p, x\, n-m+p)$ if $p > m$ and $ea = (m, x\, n)$ for $m > p$
But observe that $(ea)^*$ is the idempotent of the class containing a and so,
$(ea)^* = (m,1,m)$ and therefore
$a(ea)^* = (m,x,n)(m,1,m) = [m-n+t, x, t] = (m,x,n)$, and $a(ea)^* = ea$

Similarly,
$ae = (ae)^+ a$. Thus, S is an ample ω-semigroup.

But $\forall\, e_n \in E_S \subseteq S$, then $e_m \geq e_n$, if $n > m$, then S is also an ample ω-semigroup.

Recall that $a, b \in S$ are regular if $aba = a$ and $bab = a$. Now if $a = (m, x, n)$ and $b = (p, y, q)$, we have:
$(m, x, n)(p, y, n)(m, x, n) = (m, x, n)$ and so
$[m - n + t, xy, q - p + t](m, x, n) = (m, x, n), t = \max(n, p)$ therefore
$[m - n - q + p + w, xyx, n - q + p + t + w] = (m, x, n)$, where, $w = \max(q - p + t, m)$.
if this holds then:
$xyx = x$ and by working the other coordinates, then $t = n$.
Similarly
$(p, y, q)(m, x, n)(p, y, q) = (p, y, q)$ so that
$[p - q - n + m + u, yxy, q - n + m - v + u] = (p, y, q)$ where $v = \max(q, m)$
And $u = \max(n - m + v, p)$, so by comparing the outer coordinates and solving them out we have
$yxy = y$ and $v = q$ thus, T must be regular.
However, we can see from here that $D^* \neq \widetilde{D}$ since S has $d = 2$ D^* equivalence classes, while $\widetilde{D}$ has 4 equivalence classes of the form:
$\widetilde{D} = \{(m, x, n) \in S : m, n \in N, m \equiv a(\mathrm{mod}2), n \equiv b(\mathrm{mod}2)\}$ *and* $a \in \{0,1\} \in b$.

Conclusion

[3] constructed generalized class of $*$- bisimple ample ω-semigroups and characterized it as an extension of binary array of bisystems of cancellative monoids while the properties were given in [4]. Here we further generalize the properties of this class of semigroups as were seen in lemmas 2.1 via 2.8. In the last section, we construct a typical example of a semigroup in this class and the properties were studied.

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