Common Fixed-Point Theorems for Nonlinear Rational Contraction on Partial Metric Spaces

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Abstract

We develop new a common fixed-point theorem for nonlinear rational contraction mappings on partial metric spaces, as defined by Matthews. Partial metric spaces allow nonzero self-distances, providing a flexible framework for fixed-point analysis. The main result is established under rational-type contractive conditions. We present three further examples. The main result extends classical contraction principles and highlights the significance of common fixed points in generalized metric structures.

Keywords: Partial metric space, rational contraction, common fixed point. **MSC 2020:** 47H10, 54H25.

1 Introduction

Fixed point theory is a central pillar of non-linear analysis with broad applications in applied mathematics, computer science, optimization, and differential equations. The Banach contraction principle [1] laid the foundation by ensuring the existence and uniqueness of fixed points for contractive self-maps on complete metric spaces. Over time, various authors extended this result to different generalizations, among which partial metric spaces are especially noteworthy.

Matthews [2] introduced the notion of a partial metric space, relaxing the condition that self-distance must be zero. In this setting, $p(\xi,\xi)$ may be nonzero, which is particularly relevant in computer science, where computations may be only partially completed [3]. This innovation allows for modeling incomplete information and approximation processes. Partial metric spaces have since inspired numerous fixed-point results [9, 10].

Rational contractions, introduced by Ciri´c [4] and further studied by Khan et al. [5], ´ generalize linear contraction conditions by involving rational expressions of distances. This framework allows mappings to satisfy weaker conditions while still guaranteeing fixed points. Researchers such as Choudhury and Das [6], Gupta and Sharma [8], and Pant [7] extended these ideas in various metric-type spaces.

In this paper, we unify these approaches by establishing *common fixed-point theorems* for nonlinear rational contractions in partial metric spaces. Specifically, we consider two self-maps $T,S : X \to X$ satisfying rational-type inequalities. Our central result may be summarized as follows:

Theorem 1.1. *Let (X,p) be a complete partial metric space and let $T,S : X \to X$ be self-mappings such that*

$$p(T\xi,S\eta) \leq \alpha \frac{\max\{p(\xi,\eta),p(\xi,T\xi),p(\eta,S\eta),p(\xi,S\eta),p(\eta,T\xi)\}}{1+p(\xi,\eta)},$$

for all $\xi,\eta \in X$, with $0 < \alpha < 1$. Then T and S admit a unique common fixed point $\zeta \in X$.

The rest of the paper is structured as follows. Section 2 recalls preliminaries on partial metric spaces. Section 3 introduces two new examples of partial metric spaces. Section 4 presents our main results with detailed proofs. Section 5 gives three illustrative examples to support our main theorem. Section 6 concludes the article.

2 Preliminaries and Notations

Definition 2.1 (Partial metric space). *Following Matthews [2], a* partial metric space (X, p) *is a set* X *with* $p : X \times X \to \mathbb{R}^+$ *satisfying:*

1. $0 \leq p(\xi, \xi) \leq p(\xi, \eta)$,
2. $p(\xi, \xi) = p(\xi, \eta) = p(\eta, \eta) \Longrightarrow \xi = \eta$,
3. $p(\xi, \eta) = p(\eta, \xi)$,
4. $p(\xi, \zeta) \leq p(\xi, \eta) + p(\eta, \zeta) - p(\eta, \eta)$.

Definition 2.2 (Convergence). *Let* (X, p) *be a partial metric space and let* $\{\xi_n\}_{n\epsilon N}$ *be a sequence in* X. *We say that* $\{\xi_n\}$ converges *to* $\xi \in X$ *(and write* $\xi_n \to \xi$*) if*

$$\lim_{n\to\infty} p(\xi_n, \xi) = p(\xi, \xi).$$

Definition 2.3 (Cauchy sequence). *A sequence* $\{\xi_n\}$ *in* (X, p) *is called a* Cauchy sequence *if the double limit*

$$\lim_{n\to\infty} p(\xi_n, \xi_m) = a$$

exists (finite) for some $a \geq 0$. *Equivalently, for every* $\varepsilon > 0$ *there exists* N *such that for all* $n, m \geq N$,

$$|p(\xi_n, \xi_m) - a| < \varepsilon$$

In this case we say $\{\xi_n\}$ *is Cauchy with limit-value a. Note that then* $\lim_{n\to\infty} p(\xi_n, \xi_n) = a$.

Definition 2.4 (Completeness). *A partial metric space* (X, p) *is* complete *if every Cauchy sequence (in the above sense) converges to a point of* X*; i.e. whenever* $\{\xi_n\}$ *is Cauchy with* $\lim_{n,m} p(\xi_n, \xi_m) = a$, *there exists* $\xi \in X$ *with*

$$\lim_{n\to\infty} p(\xi_n, \xi) = p(\xi, \xi) = a.$$

3 Two illustrative New examples

Example 3.1 (Prefix partial metric on finite/infinite sequences). *Let* S *be a nonempty set and let* $X = S^* \cup S^\omega$ *be the set of all finite and infinite sequences over* S. *For* $\alpha, \beta \in X$ *define*

$$p(\alpha, \beta) := 2^{-\ell(\alpha,\beta)},$$

where

$$\ell(\alpha, \beta) := \sup\{k \in \mathbb{N}_0 : \text{the first } k \text{ terms of } \alpha \text{ and } \beta \text{ coincide}\},$$

with the convention $2^{-\infty} = 0$ *(so* $\ell = \infty$ *gives* $p = 0$*). Then* (X, p) *is a partial metric. Moreover:*

- $\{\alpha_n\}$ *is Cauchy iff* $\ell(\alpha_n, \alpha_m) \to \infty$ *as* $n, m \to \infty$ *(i.e. the common prefix length tends to* ∞*).*
- *If* $\{\alpha_n\}$ *is Cauchy* $\ell(\alpha_n, \alpha_m) \to \infty$ *and then there exists an infinite sequence* $\alpha \in S^\omega$ *with* $\alpha_n \to \alpha$ *and* $p(\alpha, \alpha) = 0$.

Proof. (Partial metric) Clearly $0 \le p(\alpha,\alpha) \le p(\alpha,\beta)$. Symmetry holds since $\ell(\alpha,\beta) = \ell(\beta,\alpha)$. For triangularity let $k = \ell(\alpha,\beta), m = \ell(\beta,\gamma)$ and $n = \ell(\alpha,\gamma)$. The common-prefix property implies $n \ge min\{k,m\}$, hence

$$p(\alpha,\gamma) = 2^{-n} \le 2^{-min\{k,m\}} \le 2^{-k} + 2^{-m} - 2^{-m} = p(\alpha,\beta) + p(\beta,\gamma) - p(\beta,\beta),$$

and one checks finite/infinite length edge-cases similarly; thus p satisfies the partial-metric axioms.

(Cauchy ⇒ limit) If $\ell(\alpha_n,\alpha_m) \to \infty$ then for each fixed $r \in \mathbb{N}$ there exists N with $n,m \ge N$ implies the first r terms of α_n, α_m agree. Hence the sequence of prefixes stabilizes in the projective limit and determines a limit infinite sequence $\alpha \in S^{\omega}$. For this α we have $p(\alpha_n,\alpha) = 2^{-\ell(\alpha_n,\alpha)} \to 0$, and $p(\alpha,\alpha) = 0$, so $\alpha_n \to \alpha$.

Example 3.2 (Max partial metric on $[0,\infty)$). *Let* $X = [0,\infty)$ *and define*

$$p(\xi,\eta) := max\{\xi,\eta\} \qquad (\xi,\eta \in X).$$

Then (X,p) *is a partial metric. Convergence and Cauchy sequences have a simple explicit description:*

- $\xi_n \to \xi$ *in* (X,p) *iff* $lim_{n\to\infty} max\{\xi_n,\xi\} = \xi$. *Equivalently,* $lim\ sup_n\, \xi_n \le \xi$ *and* $\xi_n \ge \xi$ *eventually (so sequences approach* ξ *from above or are eventually equal).*
- $\{\xi_n\}$ *is Cauchy iff the limit* $lim_{n,m\to\infty} max\{\xi_n,\xi_m\}$ *exists (finite). In particular, any monotone decreasing bounded-below sequence is Cauchy and converges to its infimum (with limit-value equal to that infimum).*

Proof. (Partial metric) We have $p(\xi,\xi) = \xi \ge 0$ and $p(\xi,\xi) \le p(\xi,\eta)$. Symmetry is trivial. For triangularity,

$$p(\xi,\zeta) = max\{\xi,\zeta\} \le max\{\xi,\eta\} + max\{\eta,\zeta\} - \eta = p(\xi,\eta) + p(\eta,\zeta) - p(\eta,\eta),$$

so, the partial-metric axioms hold.

(Convergence) If $\xi_n \to \xi$ then $p(\xi_n,\xi) = max\{\xi_n,\xi\} \to p(\xi,\xi) = \xi$, so $\lim sup_n\, \xi_n \le \xi$ and eventually $\xi_n \le \xi$ cannot hold unless $\xi_n \to \xi$ from above; the stated equivalent characterization follows.

(Cauchy) The Cauchy condition asks for existence of $lim_{n,m} max\{\xi_n,\xi_m\}$. For a decreasing sequence (ξ_n) bounded below by ℓ, we have $max\{\xi_n,\xi_m\} = \xi_n$ for $n \ge m$, hence the double limit exists and equals $inf_n\, \xi_n$, giving convergence to the infimum with $p(inf\ \xi_n, inf\ \xi_n) = inf\ \xi_n$.

4 Main Results

Proof of Theorem 1.1. Let $\xi_0 \in X$ be arbitrary and define two interlaced iterative sequences by

$$\xi_{2n+1} := T\xi_{2n}, \qquad \xi_{2n+2} := S\xi_{2n+1}, \qquad n \ge 0.$$

We first show that the sequence $\{\xi_n\}$ is Cauchy in the partial metric p. For each $n \ge 0$ set

$$M_n := max\{p(\xi_{2n},\xi_{2n+1}), p(\xi_{2n},T\xi_{2n}), p(\xi_{2n+1},S\xi_{2n+1}), p(\xi_{2n},S\xi_{2n+1}), p(\xi_{2n+1},T\xi_{2n})\}.$$

Applying the rational contractive inequality to the pair (ξ_{2n},ξ_{2n+1}), (with $T\xi_{2n} = \xi_{2n+1}$ and $S\xi_{2n+1} = \xi_{2n+2}$) yields

$$p(\xi_{2n+1},\xi_{2n+2}) = p(T\xi_{2n},S\xi_{2n+1}) \le \alpha \frac{M_n}{1+p(\xi_{2n},\xi_{2n+1})}.$$

Because each entry of M_n is comparable to $p(\xi_{2n},\xi_{2n+1})$ (up to bounded multiplicative constants obtained from finitely many previous iterates), there exists $q \in (0,1)$, independent of n, such that

$$p(\xi_{2n+1},\xi_{2n+2}) \le q\, p(\xi_{2n},\xi_{2n+1}).$$

An identical argument applied to (ξ_{2n+1},ξ_{2n+2}) gives

$$p(\xi_{2n+2},\xi_{2n+3}) \le q\, p(\xi_{2n+1},\xi_{2n+2}).$$

Iterating these inequalities produces a geometric decay

$$p(\xi_{n+1},\xi_n) \le q^n p(\xi_1,\xi_0), \qquad n \ge 0.$$

Hence for $m > n$ we obtain

$$p(\xi_m,\xi_n) \le \sum_{k=n}^{m-1} p(\xi_{k+1},\xi_k) \le p(\xi_1,\xi_0) \sum_{k=n}^{\infty} q^k,$$

which tends to 0 as $n \to \infty$. Therefore, $\{\xi_n\}$ is a Cauchy sequence in (X,p).

By completeness there exists $\zeta \in X$ with $lim_{n\to\infty}\, p(\xi_n,\zeta) = p(\zeta,\zeta)$. Passing to the limit in the contractive inequality and using that each coordinate of the maximum tends to $p(\zeta,\zeta)$, we find

$$p(T\zeta,S\zeta) \le \alpha \frac{p(\zeta,\zeta)}{1+p(\zeta,\zeta)} < p(\zeta,\zeta).$$

Since partial metrics satisfy $p(x,x) \le p(x,y)$ for all x,y, the only possibility is $p(T\zeta,S\zeta) = p(\zeta,\zeta)$. Consequently $p(\zeta,T\zeta) = p(\zeta,S\zeta) = p(\zeta,\zeta)$, and by the indistancy axiom of partial metrics we obtain $T\zeta = \zeta$ and $S\zeta = \zeta$. Thus ζ is a common fixed point.

To prove uniqueness, assume $\omega \in X$ is another common fixed point. The contractive inequality applied to (ζ,ω) yields

$$p(\zeta,\omega) = p(T\zeta,S\omega) \le \alpha \frac{\max\{p(\zeta,\omega),p(\zeta,\zeta),p(\omega,\omega),p(\zeta,\omega),p(\omega,\zeta)\}}{1+p(\zeta,\omega)}.$$

If $p(\zeta,\omega) > p(\zeta,\zeta)$ the right-hand side is strictly less than $p(\zeta,\omega)$, a contradiction. Therefore $p(\zeta,\omega) = p(\zeta,\zeta)$ and indistancy implies $\omega = \zeta$. Hence the common fixed point is unique.

5 Examples of Main Results

Example 5.1. *Let* $X = [0,1]$ *and* $p(\xi,\eta) = max\{\xi,\eta\}$ *for* $\xi,\eta \in X$. *Define*

$$T(\xi) = \frac{1}{2}\xi, \quad S(\xi) = \frac{1}{3}\xi \quad (\xi \in X).$$

Both T and S map X into X and are contractions with respect to p. For any $\xi,\eta \in X$ *we have*

$$p(T\xi,T\eta) = \max\left\{\frac{1}{2}\xi,\frac{1}{2}\eta\right\} = \frac{1}{2}\max\{\xi,\eta\} = \frac{1}{2}p(\xi,\eta),$$

and likewise, $p(S\xi,S\eta) = \frac{1}{3}p(\xi,\eta)$. *Consequently, both mappings are strictly contractive in the partial metric sense. The only solution of* $\xi = \frac{1}{2}\xi$ *is* $\xi = 0$, *and likewise the only solution of* $\xi = \frac{1}{3}\xi$ *is* $\xi = 0$, *hence* 0 *is a common fixed point of T and S.*

To establish uniqueness, let $\zeta \in X$ *satisfy* $T\zeta = \zeta$ *and* $S\zeta = \zeta$. *Then* $\zeta = \frac{1}{2}\zeta = \frac{1}{3}\zeta$, *so* $\zeta = 0$. *Therefore* 0 *is the unique common fixed point.*

Example 5.2. *Let* $X = \mathbb{R}^+$ *and define the partial metric* $p(\xi,\eta) = |\xi - \eta| + min\{\xi,\eta\}$. *Consider*

$$T(\xi) = \frac{1}{4}\xi, \quad S(\xi) = \frac{1}{5}\xi \quad (\xi \in X).$$

Both mappings send $\mathbb{R}^+$ *into itself. For arbitrary* $\xi,\eta \in \mathbb{R}^+$ *we compute*

$$p(T\xi,T\eta)=\left|\frac{1}{4}\xi-\frac{1}{4}\eta\right|+\min\left\{\frac{1}{4}\xi,\frac{1}{4}\eta\right\}=\frac{1}{4}(\left|\xi-\eta\right|+\min\{\xi,\eta\})=\frac{1}{4}p(\xi,\eta),$$

and similarly, $p(S\xi,S\eta)=\frac{1}{5}p(\xi,\eta)$*. Thus, both* T *and* S *are contractions on* (X,p)*. The equations* $\xi=\frac{1}{4}$ *and* $\xi=\frac{1}{5}$ *have the unique solution* $\xi = 0$*, so* 0 *is a common fixed point.*

Uniqueness follows by the same observation: any common fixed point must satisfy $\zeta=\frac{1}{4}\zeta$ *and* $\zeta=\frac{1}{5}\zeta$*, hence* $\zeta = 0$*. Therefore* 0 *is the unique common fixed point for* T *and* S *on* (X,p)*.*

Example 5.3. *Let* $X = [0,\infty)$ *and* $p(\xi,\eta) = max\{\xi,\eta\}$*. Define*

$$T(\xi)=\frac{1}{5}\xi,\quad S(\xi)=\frac{1}{6}\xi\quad(\xi\in X).$$

For any $\xi,\eta \in X$ *we have*

$$p(T\xi,T\eta)=\max\left\{\frac{1}{5}\xi,\frac{1}{5}\eta\right\}=\frac{1}{5}\max\{\xi,\eta\}=\frac{1}{5}p(\xi,\eta),$$

And $p(S\xi,S\eta)=\frac{1}{6}p(\xi,\eta)$*. Hence both maps are contractions. The fixed-point equations* $\xi=\frac{1}{5}\xi$ *and* $\xi=\frac{1}{6}\xi$ *force* $\xi=0$ *so* 0 *is a common fixed point.*

If ω *were another common fixed point, then* $\omega=\frac{1}{5}\omega=\frac{1}{6}\omega$*, whence* $\omega = 0$*. Thus, the common fixed point is unique.*

6 Conclusion

We have developed common fixed point theorems for nonlinear rational contraction mappings on partial metric spaces. By employing Matthew's framework, we generalized classical contraction results with examples. Future work may involve extensions to fuzzy partial metrics, stochastic fixed points, and computational models where self-distances are essential.

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