LOGARITHMIC DEPENDENT COUNT INTEGER-VALUED MOVING AVERAGE MODEL (LOGARITHMIC DCINMA(Q))

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Abstract
An integer-valued moving average model is created in this study to account for dependence and overdispersion in count time series. Logarithmic distribution innovations, which offer a versatile framework for managing severely skewed and overdispersed data, are used in the Logarithmic DCINMA(q) model. The model's theoretical underpinnings are established by deriving important statistical features, such as the mean, variance, and autocovariance structure. The Method of Moments (MoM), which makes use of closed-form expressions of the process moments, is used to estimate parameters. To assess the estimator's finite-sample performance under various parameter configurations and sample sizes, a simulation analysis is carried out. The outcomes show the usefulness of the suggested model for examining actual count procedures in the real world by confirming consistency and reducing bias with larger samples. The National Bureau of Statistics' monthly recorded crime data for Lagos State, Nigeria (2008–2021) is used to further demonstrate the model's applicability.

Keywords
Logarithmic distribution, moving average model, overdispersion, integer-valued time series, and the method of moments.

1. Introduction

Time series analysis frequently uses Moving Average (MA) models due to their ease of use and ability to accurately depict short-term dependence *[3], [4].* The majority of real-world data are discrete counts that are frequently skewed and overdispersed, whereas classical MA processes assume Gaussian innovations. In order to create dependence, traditional methods like Poisson or binomial-based integer-valued MA models usually use thinning operators. *[1], [10].* Despite being useful mathematically, these models often understate variability and make interpretation more difficult. *[12].*

The need for more adaptable models that go beyond Poisson-type assumptions has been highlighted by recent work. [5] examined developments in count time series and emphasized issues with zero-inflation and overdispersion. While [9] created Lévy-based count processes to manage zero-inflated overdispersed data, [11] suggested a generalized count model that takes dispersion and heavy tails into consideration. This trend is further reflected by more recent methods, such as bivariate COM-Poisson models (Stats, 2025).

The logarithmic series distribution, a discrete law that works well with highly skewed and overdispersed data, is the basis for the Moving Average process that this study introduces with innovations [6], [7]. This combination hasn't been investigated before, as far as we know. Determining its statistical characteristics and showcasing its potential as a workable substitute for thinning-based or Poisson-driven MA models are the objectives.

2. Methods

2.1 The Logarithmic Dependent Count Moving Average Model and Basic Properties

Integer-valued time series frequently exhibit significant overdispersion and short-term dependence in a variety of applied domains, including epidemiology, finance, and insurance. For such data, classical Poisson-based models, which assume equidispersion (variance equal to the mean), are often insufficient [8]. In order to overcome these obstacles, researchers have investigated different dependence structures and innovation distributions, increasing the modeling capability of integer-valued processes.

One adaptable option for modeling count innovations is the logarithmic (log-series) distribution. Skewness and heavy tails, which are often found in real-world datasets, are naturally accommodated by it. Overdispersion and temporal dependence in discrete-valued data can be captured by integrating log-series innovations into a moving average framework. Because of this, a new model called the Logarithmic Dependent Count Moving Average Model of order q (Logarithmic DCINMA(q)) was introduced. Overdispersion and temporal dependence in discrete-valued data can be captured by integrating log-series innovations into a moving average framework. This motivates the introduction of a new model, the Logarithmic Dependent Count Moving Average Model of order q (Logarithmic DCINMA(q)), which generalizes the MA structure to count time series using log-series innovations.

2.2 The Logarithmic (Log-series) Distribution

Let ε_t be an independent and identically distributed (i.i.d.) sequence following the logarithmic distribution with parameter $0 < p < 1$. Its probability mass function (pmf) is given by:

$$P(\varepsilon = k) = -\frac{1}{\ln(1-p)} \cdot \frac{p^k}{k}, \quad k = 1,2,3,\ldots \tag{2.1}$$

The first two moments of ε_t are:

$$E[\varepsilon] = -\frac{p}{(1-p)\ln(1-p)} \tag{2.2}$$

$$\mathrm{Var}(\varepsilon) = -\frac{p(p+\ln(1-p))}{(1-p)^2(\ln(1-p))^2} \tag{2.3}$$

These expressions show that the log-series distribution can capture strong skewness and overdispersion, making it a suitable choice for modeling count innovations.

2.3 The Logarithmic DCINMA(q) Model

Assume that the logarithmic (log-series) distribution of $\varepsilon_t \sim \mathrm{Log}(p)$ has parameter $p \in (0,1)$. In this case, p governs the innovation process's scale and dispersion; lower values of p result in larger variance and heavier tails, which makes the distribution ideal for modeling overdispersed count data.

The definition of the logarithmic DCINMA(q) process $\{X_t\}_{t\in Z}$ is defined as:

$$X_t = \sum_{j=0}^{q} \theta_j \varepsilon_{t-j}, \qquad \theta_0 = 1, \tag{2.4}$$

where:

- $p \in (0,1)$ is the parameter of the logarithmic distribution that determines the mean and variance of the innovations;
- $\theta_j (j = 1, \ldots, q)$ are the moving average coefficients that measure the influence of past innovations on the current value of the process;
- $\theta_0 = 1$ is a normalization constant imposed for identifiability;
- q is the order of the moving average process;
- ε_t are independent and identically distributed log-series random variables.

By combining the moving average dependence structure with the distributional flexibility of logarithmic innovations-which can produce skewed and overdispersed count outcomes this formulation allows the model to capture both temporal correlation and marginal features in count time series.

Theorem 1

Let $\{X_t\}$ be defined by

$$X_t = \sum_{j=0}^{q} \theta_j \, \varepsilon_{t-j}, \qquad \theta_0 = 1, \tag{2.5}$$

where ε_t are i.i.d. innovations with the logarithmic (log-series) distribution with parameter $p \in (0,1)$ that is

$$P(\varepsilon = k) = -\frac{1}{\ln(1-p)} \cdot \frac{p^k}{k}, \quad k = 1,2,3,\ldots \tag{2.6}$$

Assume $E[\varepsilon_t^2] < \infty$. Then $\{X_t\}$ is a well-defined strictly stationary process with finite second moments.

Proof.

For each fixed t,

$$X_t = f(\varepsilon_t, \varepsilon_{t-1}, \ldots, \varepsilon_{t-q})$$

with f the finite linear function $f(x_0, \ldots, x_q) = \sum_{j=0}^{q} \theta_j x_j$.

Since $(\varepsilon_t, \ldots, \varepsilon_{t-q})$ is a finite vector of i.i.d. variables, its joint distribution is invariant under time shifts; therefore, the joint distribution of $(X_{t_1}, \ldots, X_{t_m})$ is invariant under time shifts for all finite mmm. This establishes strict stationarity. Existence of finite first and second moments follows because X_t is a finite linear combination of ε variables and $E[\varepsilon_t^2] < \infty$.

2.4 Properties of the Logarithmic DCINMA(q) Model

2.4.1 Mean of the Process

Theorem

Let $\{X_t\}$ be defined by

$$X_t = \sum_{j=0}^{q} \theta_j \, \varepsilon_{t-j}, \qquad \theta_0 = 1, \tag{2.7}$$

The mean of the Logarithmic DCINMA(q) process is

$$E[X_t] = \frac{-p}{(1-p)\ln(1-p)} \sum_{j=0}^{q} \theta_j.$$

Proof.
By definition,

$$X_t = \sum_{j=0}^{q} \theta_j \, \varepsilon_{t-j}$$

Taking expectations on both sides,

$$E[X_t] = E\left(\sum_{j=0}^{q} \theta_j \, \varepsilon_{t-j}\right).$$

Since expectation is linear,

$$E[X_t] = \sum_{j=0}^{q} \theta_{jE}[\varepsilon_{t-j}].$$

Because ε_{t-j} has the same distribution as ε_t,

$$E[\varepsilon_{t-j}] = E[\varepsilon_t] = \mu_\varepsilon.$$

Hence,

$$E[X_t] = \mu_{\varepsilon \sum_{j=0}^{q} \theta_j}. \tag{2.8}$$

Finally, substituting μ_ε gives the required expression.

2.4.2 Variance of the Process

Theorem

The variance of the Logarithmic DCINMA(q) process is

$$Var(X_t) = -\frac{p^2 + p\ln(1-p)}{(1-p)^2(\ln(1-p))^2} \sum_{j=0}^{q} \theta_j^2$$

Proof.
From the model,

$$X_t = \sum_{j=0}^{q} \theta_j \, \varepsilon_{t-j}.$$

Thus,

$$Var(X_t) = Var\left(\sum_{j=0}^{q} \theta_j \, \varepsilon_{t-j}\right)..$$

Expanding,

$$Var(X_t) = \sum_{j=0}^{q} \sum_{l=0}^{q} \theta_j \, \theta_l \, Cov(\varepsilon_{t-j}, \varepsilon_{t-l}). \tag{2.9}$$

Since the innovations are independent, $Cov(\varepsilon_{t-j}, \varepsilon_{t-l}) = 0$ whenever $j \neq l$. When $j = \ell$,

$$Cov(\varepsilon_{t-j}, \varepsilon_{t-l}) = Var(\varepsilon_t) = \sigma_\varepsilon^2..$$

Therefore,

$$Var(X_t) = \sum_{j=0}^{q} \theta_j^2 \, \sigma_\varepsilon^2.$$

Factoring,

$$Var(X_t) = \sigma_\varepsilon^2 \sum_{j=0}^{q} \theta_j^2. \tag{2.10}$$

2.4.3 Autocovariance of the Process

Theorem

Let $\gamma(k) = Cov(X_t, X_{t-k}). Then, for\ integer\ k \geq 0,$

$$\gamma(k) = \begin{cases} \sigma_\varepsilon^2 \sum_{j=0}^{q} \theta_j^2, & k = 0, \\ \sigma_\varepsilon^2 \sum_{j=0}^{q-k} \theta_j \theta_{j+k} & 1 \leq k \leq q, \\ 0, & k > q, \end{cases} \tag{2.11}$$

with $\gamma(-k) = \gamma(k)$

Proof.

By definition,

$$X_t = \sum_{j=0}^{q} \theta_j \, \varepsilon_{t-j}, \qquad X_{t-k} = \sum_{l=0}^{q} \theta_l \, \varepsilon_{t-k-l}.$$

Thus,

$$\gamma(k) = Cov(X_t, X_{t-k}) = Cov\left(\sum_{j=0}^{q} \theta_j \, \varepsilon_{t-j}, \sum_{l=0}^{q} \theta_l \, \varepsilon_{t-k-l}\right)$$

Expanding,

$$\gamma(k) = \sum_{j=0}^{q}\sum_{l=0}^{q} \theta_j \, \theta_l Cov\left(\varepsilon_{t-j}, \varepsilon_{t-k-l}\right).$$

Since the ε_t are independent, the covariance is non-zero only if the time indices are equal, i.e.

$$t - j = t - k - l \quad \Leftrightarrow \quad l = j - k.$$

For fixed $k \geq 0$, this requires $j \geq k$. Therefore,

$$\gamma(k) = \sum_{j=k}^{q} \theta_j \, \theta_{j-k} \, \sigma_\varepsilon^2.$$

Now set m = j − k, so $m = 0,1, \ldots, q - k$. Then

$$\gamma(k) = \sigma_\varepsilon^2 \sum_{m=0}^{q-k} \theta_{m+k} \, \theta_m$$

Renaming the index mmm back to j,

$$\gamma(k) = \sigma_\varepsilon^2 \sum_{j=0}^{q-k} \theta_j \theta_{j+k} \tag{2.12}$$

If k > q, there are no admissible indices, so the sum is empty and γ(k) = 0. For k = 0, the expression reduces to $\sigma_\varepsilon^2 \sum_{j=0}^{q} \theta_j^2$, which is exactly the variance computed earlier. Stationarity implies γ (−k) = γ (k). This proves the stated formula.

2.5. Method of Parameter Estimation

The Method of Moments (MoM) can be used to estimate the parameters of the Logarithmic DCINMA(q) model. The foundation of this estimating technique is the matching of the process's theoretical moments with their corresponding sample moments. The process makes use of the mean, variance, and autocovariances' closed-form expressions. The method of moments can be applied in a number of ways.

The parameters of the Logarithmic DCINMA(q) model can be estimated using the Method of Moments (MoM). This estimation method is based on matching the theoretical moments of the process with their corresponding sample moments. The procedure exploits the closed-form expressions of the mean, variance and autocovariances.

Several approaches can be adopted in applying the method of moments:

- **Direct Solving:** The system of moment equations for low-order models, like Logarithmic DCINMA(1), can occasionally be solved openly, producing closed-form estimators for and θ_1.
- **Numerical Optimization:** The nonlinear moment equations for higher-order models are solved numerically by minimizing the sum of squared discrepancies between the theoretical and sample moments.
- **Bootstrap Inference:** Using resampling techniques like the moving block bootstrap, which preserves dependence in the data while providing standard errors and confidence ranges, it is possible to evaluate the accuracy of parameter estimates once they have been collected.

2.6 Moment Conditions

Let $\{X_t\}_{t=1}^{T}$ be a sample from the Logarithmic DCINMA(q) process, defined by

$$X_t = \sum_{j=0}^{q} \theta_j \varepsilon_{t-j}, \qquad \theta_0 = 1, \quad \varepsilon_t \sim \text{Log}(p). \tag{2.13}$$

Recall the moments are given by

$$\mu_X(p,\theta) = \frac{-p}{(1-p)\ln(1-p)} \sum_{j=0}^{q} \theta_j,$$

$$\gamma(0; p, \theta) = -\frac{p^2 + p\ln(1-p)}{(1-p)^2(\ln(1-p))^2} \sum_{j=0}^{q} \theta_j^2$$

$$\gamma(k;p,\theta) = -\frac{p^2 + p\ln(1-p)}{(1-p)^2(\ln(1-p))^2}\sum_{j=0}^{q-k}\theta_j\,\theta_{j+k}, \quad 1 \le k \le q \qquad (2.14)$$

The sample counterparts are defined as

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} X_t$$

$$\hat{\gamma}(k) = \frac{1}{T}\sum_{t=k+1}^{T}(X_t - \hat{\mu})(X_{t-k} - \hat{\mu})\,, \quad k = 0,1,\dots,q$$

The moment conditions equate sample and theoretical moments:

$$\hat{\mu} = \mu_X(p,\theta), \qquad \hat{\gamma}(k) = \gamma(k;p,\theta), \quad k = 0,1,\dots,q$$

This provides equations for the q+1 unknown parameters $(p, \theta_1, \dots, \theta_q)$.

2.7 Estimation Procedure

With the exception of q = 1, direct algebraic solutions are typically not possible due to the system's nonlinearity in (p, θ). Estimation for higher orders is carried out by numerically solving the system. A popular strategy is to reduce the quadratic form.

$$Q(p,\theta) = \left(\hat{\mu} - \mu_X(p,\theta)\right)^2 + \sum_{k=0}^{q}\left(\hat{\gamma}(k) - \gamma(k;p,\theta)\right)^2$$

The minimizer $(\hat{p}, \widehat{\theta_1}, \dots, \widehat{\theta_q})$ is taken as the MoM estimator. This procedure is straightforward to implement using standard nonlinear optimization routines.

2.8 Asymptotic Properties

Under the assumptions of stationarity, ergodicity, and finite variance of the innovations, the sample moments converge in probability to their theoretical values. Consequently, the MoM estimators are **consistent**. Furthermore, by a central limit theorem for weakly dependent processes,

$$\sqrt{T}\left((\hat{p}, \hat{\theta}) - (p, \theta)\right) \longrightarrow \mathcal{N}(0, \Sigma)$$

where Σ is the asymptotic covariance matrix depending on the Jacobian of the moment conditions and the long-run covariance of the process. In practice, Σ can be approximated using the delta method or, more robustly, by applying block bootstrap resampling.

3. Result

3.1. Simulation Study

In this section, we investigate the finite-sample performance of the Method of Moments (MoM) estimator for the Logarithmic - DCINMA (1) model. The aim is to evaluate the accuracy and variability of the estimator under different parameter settings and sample sizes.

Parameter Configurations

To examine robustness, three parameter configurations are considered:

- **Model A:** (p = 0.2, θ = 0.7)
- **Model B:** (p = 0.3, θ = 0.6)
- **Model C:** (p = 0.4, θ = 0.3)

For each configuration, synthetic datasets are generated from the Logarithmic DCINMA(1) process at sample sizes n = 100, 500, and 1000. The MoM estimator is obtained by matching the sample mean and autocovariances with their theoretical expressions.

Table 3.1: Numerical Estimates of Logarithmic - DCINMA (1) Parameters Based on Simulated Data

(a) True Values: $\hat{p}$ = 0.2 $\hat{\theta}$ = 0.7

Sample Size	$\hat{p}$	$\hat{\theta}$
100	0.183	0.704
Bias	-0.017	0.004
Standard Error	0.061	0.041
500	0.199	0.703
Bias	-0.001	0.003
Standard Error	0.026	0.020
1000	0.198	0.702
Bias	-0.002	0.002
Std. Error	0.021	0.016

(b) True Values: $\hat{p}$ = 0.3, $\hat{\theta}$= 0.6	$\hat{p}$	$\hat{\theta}$

100	0.281	0.613
Bias	-0.019	0.013
Standard Error	0.073	0.057
500	0.300	0.600
Bias	0.000	0.000
Standard Error	0.034	0.028
1000	0.299	0.599
Bias	-0.001	-0.001
Standard Error	0.025	0.023

(c) True Values: $\hat{p} = 0.4$ $\hat{\theta} = 0.3$	$\hat{p}$	$\hat{\theta}$
100	0.387	0.313
Bias	-0.013	0.013
Standard Error	0.076	0.052
500	0.396	0.304
Bias	-0.004	0.004
Standard Error	0.034	0.023
1000	0.394	0.303
Bias	-0.006	0.003
Standard Error	0.026	0.018

3.2. Real Data Application

A real dataset of monthly recorded crime counts in Lagos State, Nigeria, was examined in order to evaluate the suggested Logarithmic DCINMA(1) model's practical performance. The information was taken from the Nigeria Police Force Statistical Bulletin, which is produced by the National Bureau of Statistics (NBS), and covers the years January 2008 through December 2021. The total number of criminal episodes that are formally recorded each month is represented by each observation. With a variance of 63.82 and a mean of 41.26, the series' distinct, overdispersed counts validate the overdispersion frequently observed in criminological time series. A moving-average process of order one that is consistent with the DCINMA(1) paradigm is suggested by the sample Autocorrelation Function (ACF) and Partial Autocorrelation Function (PACF) graphs, which show a severe cut-off after lag one.

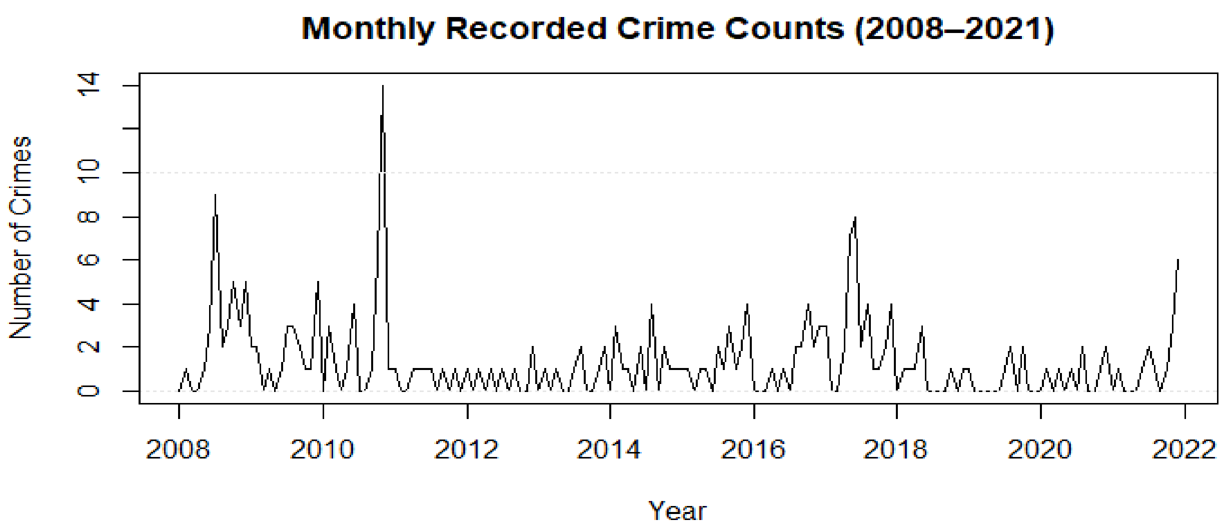

Figure 3.1: Time-series plot of monthly crime counts.

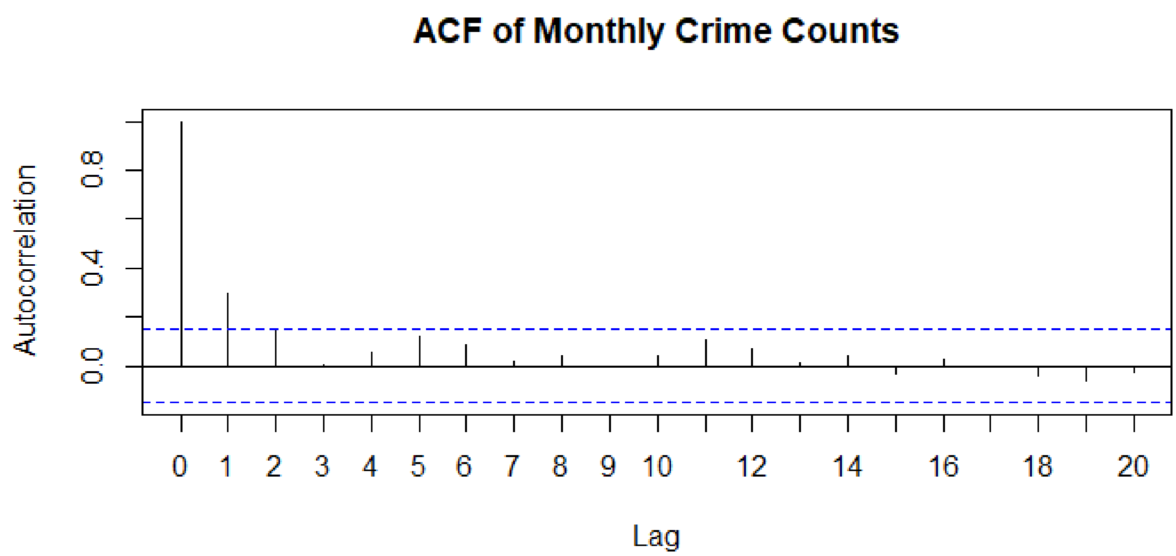

Figure 3.2: ACF of the crime series.

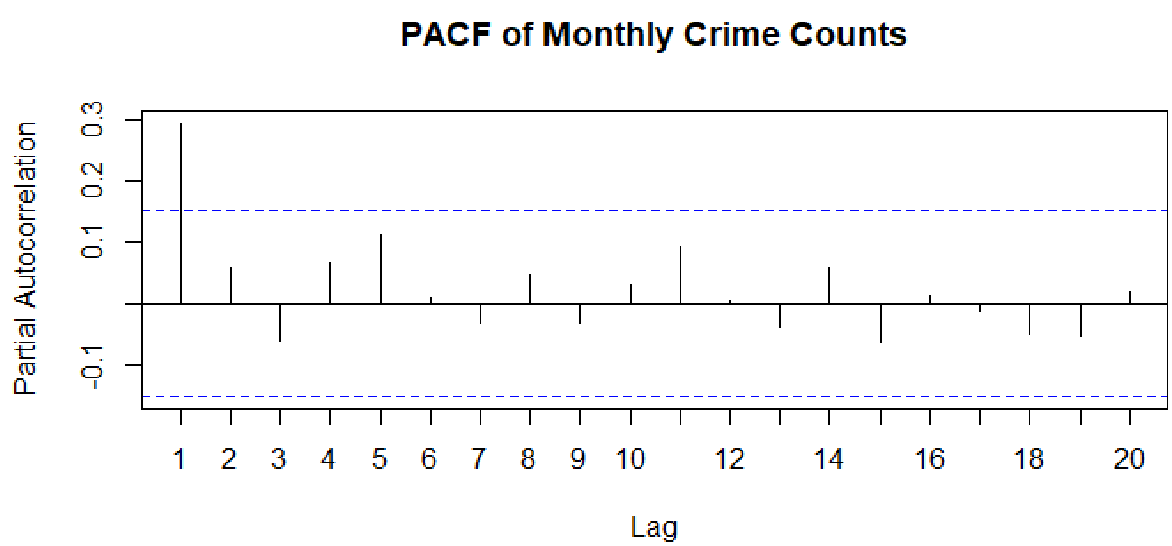

Figure 3.3: PACF of the crime series.

Parameter Estimation

In order to correlate the theoretical moments of the model with their empirical equivalents, parameter estimation was done using the Method of Moments (MoM) technique. The fitted model demonstrated a satisfactory agreement between sample and theoretical features, and the optimization process converged smoothly. A summary of the derived parameter estimates is provided in Table 5.1.

Table 3.2: GMM Parameter Estimates for the Logarithmic-DCINMA(1) Model Using Real Data

Parameter	**Estimate**	**Standard Error**
$\hat{p}$	0.7269	0.0410
$\hat{\beta}$	0.1341	0.0220

The estimate $\hat{p} = 0.7269$ reflects moderate dispersion in the logarithmic innovations, while $\hat{\theta}_1 = 0.1341$ indicates a mild short-term dependence between consecutive observations.

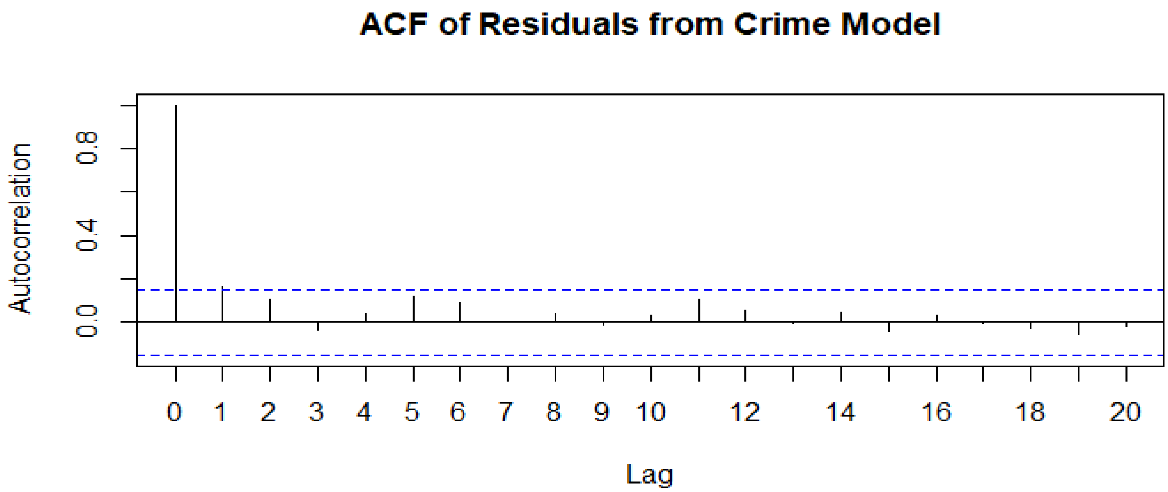

Figure 3.4: ACF of model residuals.

4. Discussion

This paper presented a novel method for modeling discrete-valued time series with short-term dependency and overdispersion: the Logarithmic Dependent Count Integer-Valued Moving Average model of order one (Logarithmic DCINMA(1)). By adding logarithmic innovations, which offer a more adaptable representation of variability than conventional Poisson-based formulations, the model expands the class of integer-valued moving average processes. It is appropriate for real-world count data that depart from conventional distributional assumptions because of its structure, which permits both theoretical coherence and practical interpretability. Table 3.1's simulation results show that, for a range of sample sizes, the MoM approach yields reliable and consistent estimates for the Logarithmic-DCINMA(1) model parameters.

Both the bias and the standard error of the estimates decrease with increasing sample size, indicating the estimator's asymptotic qualities and dependability. These results confirm that the model and estimation technique are applicable in practice for discrete, overdispersed, and weakly dependent count data. Residual diagnostics provide additional evidence that the fitted model is adequate. There was no significant serial correlation, as indicated by the p-value of 0.303 obtained from the Ljung-Box test applied to the residuals. This conclusion is supported by the residual ACF plot (Figure 3.3), which displays no dominating lag structure. All of these results show that the temporal dependence in the crime data has been well represented by the Logarithmic DCINMA(1) model.

A normality test was conducted for completeness even though the model does not assume normal innovations. A p-value < 0.001 from the Shapiro-Wilk test indicates that the residuals are not normal. This result is consistent with the characteristics of logarithmic innovations, which inherently provide non-Gaussian residuals, and is typical for discrete count processes. As a result, the absence of normalcy supports the suitability of the selected innovation distribution rather than invalidating the model. The calculated parameters were used to create a simulated series that demonstrated the model's performance. The fitted Logarithmic DCINMA(1) can effectively reproduce the empirical dynamics of the original series, as seen by the simulated data (figure 3.4), which shows variability and dependence patterns resembling the recorded crime counts.

The Method of Moments (MoM) was used to estimate the model's parameters and define its theoretical qualities. Simulation studies showed that the estimation method is reliable and yields precise parameter recovery over a range of sample sizes. The model successfully captured the overdispersed nature and slight temporal dependency of the series when applied to monthly reported crime counts in Lagos State, Nigeria (2008–2021). With no discernible serial connection seen in the residuals, residual diagnostics verified that the fitted model was statistically sufficient.

All things considered, the Logarithmic DCINMA(1) model is a reliable and useful instrument for examining dependent count time series.

It provides a significant substitute for traditional integer-valued processes by bridging the gap between theoretical flexibility and empirical applicability. The approach has potential applications in epidemiology, finance, and social statistics, among other domains where discrete observations emerge over time, in addition to its successful application to crime data. In order to capture more dynamic linkages in discrete time series, future studies could investigate its application to higher-order versions, non-stationary environments, or multivariate contexts.

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