Knowledge of fractional equivalence: A case of two 3-3-3 elementary teacher education pre-service teachers.

Dambakushamba Shorayi
Department of Natural Sciences, Mathematics Teacher Education Lecturer,

Nyadire College of Education, Mutoko, Zimbabwe
jdambah@gmail.com

Abstract

The study examined how the 3-3-3 teacher education model prepares pre-service teachers to understand fraction equivalence. Using a qualitative case study design, data were collected from two final-year elementary school pre-service teachers at a teacher education college in Zimbabwe through a 5-item test. The participants were purposively selected because they taught grade 6 during their teaching practice. Results showed that while the participants understood unit equivalence, they struggled with proportional equivalence and had difficulty manipulating fractions with differing quantities. They performed better with quantitative invariance but had difficulty comparing shapes, preferring structural invariance. The study concluded that the 3-3-3 teacher education model does not equip pre-service teachers with the knowledge of proportional equivalence. The study contributed to knowledge by highlighting the importance of conceptual understanding of fraction equivalence, a concept that has been ignored and taken for granted even in teacher education.

Keywords

unit equivalence, proportional equivalence, teacher education.

INTRODUCTION

Fractions are crucial components for achieving precision in the manipulation of quantities that do not always exist discretely [35], [42]. The elementary school mathematics syllabus in the Zimbabwean curriculum has four core concepts, namely, numbers, operations, measures, and relationships, that are taught in a spiral manner from grade one to grade seven. All four core concepts use the knowledge of fractions in one form or another from as early as grade two. Research indicates that an effective understanding of fractions is also crucial for understanding advanced mathematical reasoning and problem-solving skills [11], [48], [42], [4]. However, there is an alleged inadequate conceptual understanding of the subject of fractions that poses considerable instructional challenges for elementary pre-service teachers (PSTs) globally [19], [33], [50]. This lack of comprehension hinders the PSTs' ability to effectively teach fraction concepts to their learners when they go out on teaching practice (TP), ultimately impacting learner understanding at the formative stage.

PROBLEM STATEMENT

PSTs teach the concepts of fractions when they go out on TP. However, despite numerous studies on PSTs' knowledge of fractions, the extent to which PSTs undertaking the 3-3-3 teacher education at a teacher education college in Mashonaland East, Zimbabwe, understand the different concepts of fraction equivalence has not been assessed. Therefore, this study assessed the extent to which PSTs undertaking the 3-3-3 teacher education programme at this teacher education college understood the concept of quantitative invariance and equivalent fractions and highlighted the importance of foundational mathematical concepts in teacher education.

OBJECTIVES OF THE STUDY

The objectives of this study were:

1. to determine the extent to which elementary pre-service teachers understand the concept of fraction equivalency.
2. to explore how elementary PSTs represent fractions diagrammatically to indicate equivalence.
3. to analyse the effectiveness of the 3-3-3 elementary teacher education model in equipping pre-service teachers with the knowledge of different forms of fraction equivalence.

RESEARCH QUESTIONS

To achieve the stated research objectives, the following questions were asked:

1. To what extent do elementary school pre-service teachers understand the concept of fraction equivalency?
2. How do PSTs represent fractions diagrammatically to indicate equivalence?

3. To what extent is the 3-3-3 elementary teacher education effective in equipping pre-service teachers with the knowledge of different forms of fraction equivalency?

LITERATURE REVIEW

Elementary PSTs often maintain a fixed understanding of fractions despite completing teacher education in elementary mathematics. It is common to find a fraction chart in elementary school classrooms, grades 2 to 7, which often misrepresents the variability of the whole by repeatedly dividing a labelled referent rectangle into equal parts for various denominators, typically up to 10. This leads to misconceptions about fraction equivalence, as PSTs tend to view it merely as a symbolic manipulation rather than a size. Consequently, PSTs exhibit limited SCK regarding fraction equivalence and struggle to articulate the principle of quantitative invariance, which undermines their conceptual understanding and leads to teaching methods that emphasize memorizing procedures rather than conceptual comprehension. Moreover, [50] argue that it is necessary first to grasp the main functions of fractions, such as representing parts of a whole, comparing quantities, and performing arithmetic operations, to appreciate the concept of a fraction.

The approach used by PSTs generally misrepresents real-life spaces and contributes to a misconception that, for quantities to be comparable, they should originate from structurally invariant equal entities, that is, quantitatively invariant. Consequently, PSTs primarily recognize one form of fraction equivalence, that is, unit equivalence. This highlights a significant gap in PSTs' knowledge of fractions. Such limited knowledge can result in instructional methods that emphasize memorization rather than a solid conceptual foundation. However, according to [45], although memorization enables fluency, conceptual understanding requires productive disposition, adaptive reasoning, and strategic competence that support the growth of creative minds. This implies that overdependence on one strategy for acquiring procedures for dealing with fractions will stifle the other competencies and hinder creativity and innovation among learners. Moreover, creativity and innovation can allegedly be hampered when the teaching of mathematics remains step-by-step, where the teacher revises the previous concept, demonstrates a few examples, learners regurgitate in pairs, and then write an exercise. This implies that the development of the concept of fraction equivalence is also rigid, where PSTs only understand and teach one form of equivalence, which fixes the concept to imply 'we can only deal with equal entities.' PSTs' rigidity emanates from their ineffective understanding of the concept, which leads to poor strategies and errors when working with learners' responses during the teaching of the concept of fractions [33], [49].

This study was guided by [6] practice-based MKT theoretical framework. MKT is regarded as 'the mathematical knowledge needed to carry out the work of teaching mathematics' [7]. They identified that MKT includes the tasks involved in teaching and analysing the mathematical demands of these tasks, like guiding learners in ways to solve problems, answering learners' questions, and marking their work. Moreover, to be able to do all this, the teacher should understand the school curriculum. It draws from Shulman's two teacher knowledge domains, namely pedagogical content knowledge and subject matter knowledge. The framework defines the knowledge that teachers need for effective engagement in their teaching of mathematics. They argued that MKT must account for both regularities and uncertainties of a teacher's practice. [6] and [41] ideas are similar in that they both acknowledge the need for teachers to be knowledgeable of their subject matter, necessary to support effective learning. However, the researcher feels that Shulman's PCK and SMK are overly generalised, unlike [6] theory of MKT, which focuses specifically on the teaching of mathematics. As a result, my research falls under the MKT practice-based theory [6]. The practice-based theory contents that the PCK did not adequately address the need for MKT to prepare teachers to understand the settings of the real problems they solve when teaching mathematics on a daily basis. To clarify what teachers need to be effective, [6] expanded the SMK to include three domains: Common Content Knowledge (CCK), Horizon Content Knowledge (HCK), and Specialized Content Knowledge (SCK). PCK was divided into three categories: Knowledge of Content and Curriculum, Knowledge of Content and Students, and Knowledge of Content and Teaching.

SCK is described as "the mathematical knowledge and skill specific to the activity of teaching." In contrast, Common Content Knowledge (CCK) is "the mathematical knowledge and skills employed in situations other than teaching" [6]. Research has it that HCK is "the knowledge of how the mathematical topics are interrelated with the scope and sequence of school mathematics" [46]. SMK is perceived to have a notable impact on pre-service teachers' effectiveness in teaching. Mathematical teaching needs SCK that allows PSTs to represent concepts comprehensively and understand learners' mathematical thinking [30], [39].

This research focused on the SCK sub-domain of MKT, specifically forms of equivalence when dealing with fractions, because research has identified it as very weak among pre-service elementary school teachers [2], [5], [17], [22], [26], [43]. This observation suggests that by the time learners reach high school and higher levels of their education, they may have developed a very weak understanding of mathematics, especially in fractions, which research has also identified as a difficult concept for many [11], [19], [20], [23], [25], [29], [33], [36], [43]. Research shows that when learners fail to comprehend the concept of fractions during elementary school, they continue to struggle with it in middle and high school [23]. This suggests that a weak elementary school foundation in fractions and mathematical knowledge in general jeopardizes a learner's further understanding of the concept.

The work of [15] has shown that learners may encounter obstacles in understanding concepts associated with fractions, such as the representation, operations, and framing of rational numbers. According to these authors, the difficulties encountered in learning fractions can be attributed to various factors, including the abstract nature of these concepts, challenges related to the specific vocabulary associated with arithmetic, as well as gaps in mathematical reasoning and metacognition.

Fraction constructs

The concept of fractions takes many forms. Research identifies five, namely: part-whole, ratio, measure, operator, and quotient [15]. Comparison is one of the uses of fractions across these constructs. The part-whole construct involves partitioning a continuous quantity or a set of discrete objects into equal-sized parts [35]. When comparing quantities, the part-whole construct treats equal parts and equal wholes as quantitatively invariant. Therefore, both proportional and unit equivalences must be linked to the presence of an equal whole [31]. Otherwise, without the equal whole, one cannot compare. The ratio construct is regarded as a comparative index rather than a number [31], [35]. The quantities being compared in a ratio need not be equal, and as such, the ratio construct has proportional equivalence [31]. Under the measure construct, which is based on the principle of fixing a unit of measure, unit equivalence is the most evident form of equivalence, indicated when understanding is based on partitive division [31]. Under the operator construct, in which the fraction is a function that transforms a number, object, or set into a new quantity that is either stretched or shrunk ([15], [31], both unit equivalence and proportional equivalence exist. According to [15], the result of a division such as 24÷ 6 ($a\ two{-}entity$), which is written in the form $\frac{24}{6}$ (a one-entity), has the dividend as the numerator and the divisor as the denominator. Both unit equivalence and proportional equivalence exist in the quotient construct.

METHODOLOGY

Research Design

The study employed a case study methodology in which qualitative data were gathered and analysed. The case study examined cases of content from participants who responded to questions about their knowledge of forms of fraction equivalence [9], [14], [16]. “Content analysis entails a systematic reading of a body of texts, images, and symbolic matter, not necessarily from an author’s or user’s perspective” [21].

Research Site and Research Participants

Participants in this study were purposively sampled from twenty-two pre-service teachers who were in the final year of the 3-3-3 teacher education program at an elementary school teacher education college in Mashonaland, Zimbabwe. The college was purposively sampled from fourteen elementary school teacher education colleges in Zimbabwe after obtaining permission to conduct research in teacher education colleges from the Ministry of Higher and Tertiary Education, Innovation, Science and Technology Development (see Appendix B) during my study for a PhD with the University of South Africa which gave me the Ethics Approval Ref. 2023/07/05/16693353/08/AM, which expires on 05/07/2028. This case study researcher used purposive sampling because the aim was to evaluate elementary school pre-service teachers’ knowledge of fraction equivalence by drawing on the wealth of information held by finalists who taught Grade 6 during teaching practice [9].

Data Collection and Data Analysis

Data were collected from two pre-service teachers in their final year of the three-year 3-3-3 teacher education program using a five-item pen-and-paper equivalent fractions test. The researcher developed the data-gathering instrument and administered it to the participants, who responded to the items and submitted their responses after one hour. The participants in this study are identified as Molly and Diva. The test items assessed the pre-service teachers’ knowledge of the forms of fraction equivalence. Items 1 to 4 were open-ended questions that allowed participants to express their opinions. Item 5 used Likert-type questions, in which participants selected from five possibilities, ranging from strongly disagree (1) to strongly agree (5). Therefore, data analysis involved analysing the content to identify the different meanings that formed themes [34]. The content was coded manually, and themes were identified. These themes served as the basis for analysing the data.

FINDINGS AND DISCUSSION

Findings from the study are presented below under the appropriate research questions.

Research Question 1: To what extent do elementary school pre-service teachers’ definition of

fraction equivalency includes unit and proportional equivalences?

Findings from item 1 were used to answer this research question.

Item 1: When are fractions equivalent?

Item 1 required participants to define the phrase ‘equivalent fractions’. Participant Molly’s response was;

> ‘*Fractions are equivalent when they represent the same part of a whole, even if they have different numerators and denominators.*

Participant Molly's response indicates that her understanding of the equivalence of fractions is limited to the unit equivalence, that is, the part-whole. It implies that the participant observes quantity invariance (see Figure 1). Moreover, the researcher expected the definition to be more encompassing, implying the definition should indicate that equivalence is not limited to unit equivalence.

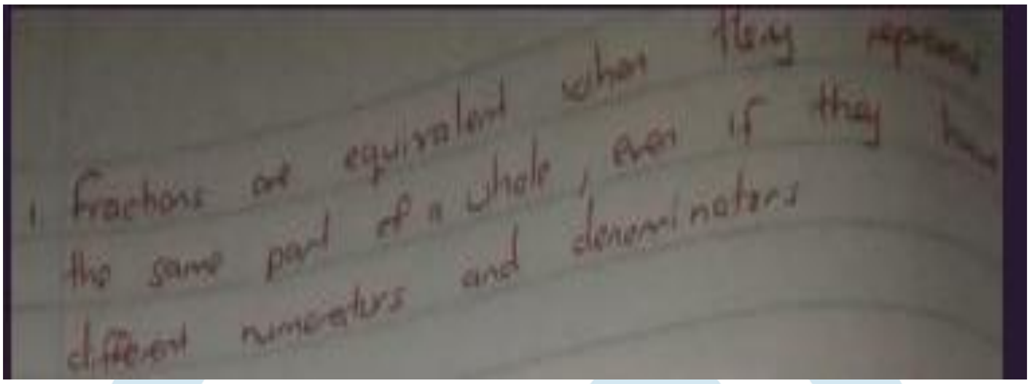


Fig. 1: Molly's response to item 1

Participant Diva explains the equivalence of fractions as,

> *'Fractions are equivalent when they represent the same value or the same part of a whole, even if their numerators and denominators are different.*

Like in participant Molly's definition, participant Diva is limited to quantitative invariance where he thought of the same part of a whole, implying unit equivalence (see Figure 2).

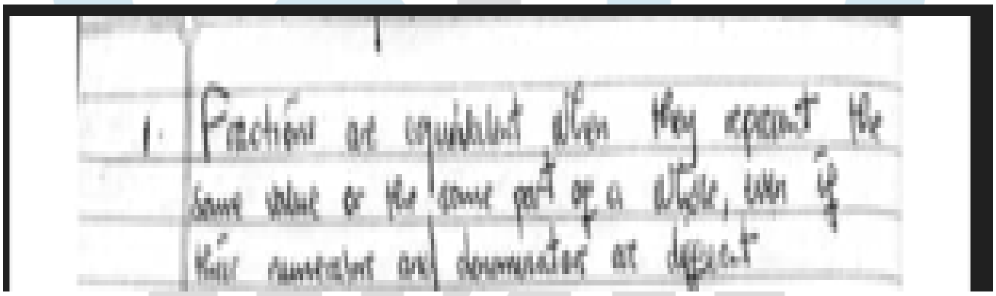


Fig. 2: Diva's response to item 1

The two definitions reveal that both participants' knowledge of equivalent fractions was limited unit equivalence.

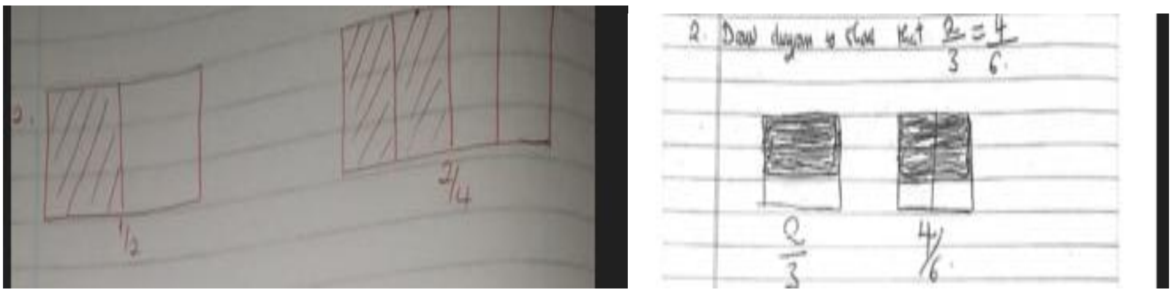
Research Question 2: How do PSTs represent fractions diagrammatically to determine

equivalence?

Items 2 and 5 were used to answer this question. Participants were asked to represent given fractions on diagrams. The following excerpts are responses from the two participants.

Item 2: Draw diagrams to show that $\frac{2}{3} = \frac{4}{6}$.

Participants were required to represent the fractions $\frac{2}{3}$ and $\frac{4}{6}$ diagrammatically. Figures 4 and 5 indicate that both participants drew rectangles and used the area model in their attempts to represent the fractions. However, participant Molly changed the instruction from $\frac{2}{3}$ and $\frac{4}{6}$ to be $\frac{1}{2}$ and $\frac{2}{4}$, respectively, and represented them correctly (see Figure 3). Although changing question items would change the complexity of the question, the researcher regarded this representation as indicating that the participant managed to represent the fractions correctly. The researcher enquired whether the rectangles were equal, and participant Molly said they were, implying that the participant was aware of unit equivalence and the concept of quantitative invariance.



Molly Diva

Fig. 3: Fraction picture representations

Participant Diva drew two rectangles of the same size (quantitative invariance), initially each rectangle had three equal parts. One was then demarcated to have six equal parts, representing two parts in each third (see Figure 3). Like participant Molly, Diva used the area model to represent the fractions $\frac{2}{3}$ and $\frac{4}{6}$ on the two rectangles correctly, implying the participant was well versed with quantitative invariance in showing the equivalence of fractions. Moreover, the two participants indicated that when talking of unit equivalence, the parts should be the same size and the wholes should be equal implying invariant. Both participants indicated knowledge of representing fractions pictorially.

In item 5, participants selected from 1-strongly disagree to 5-strongly agree and briefly explained their choices.

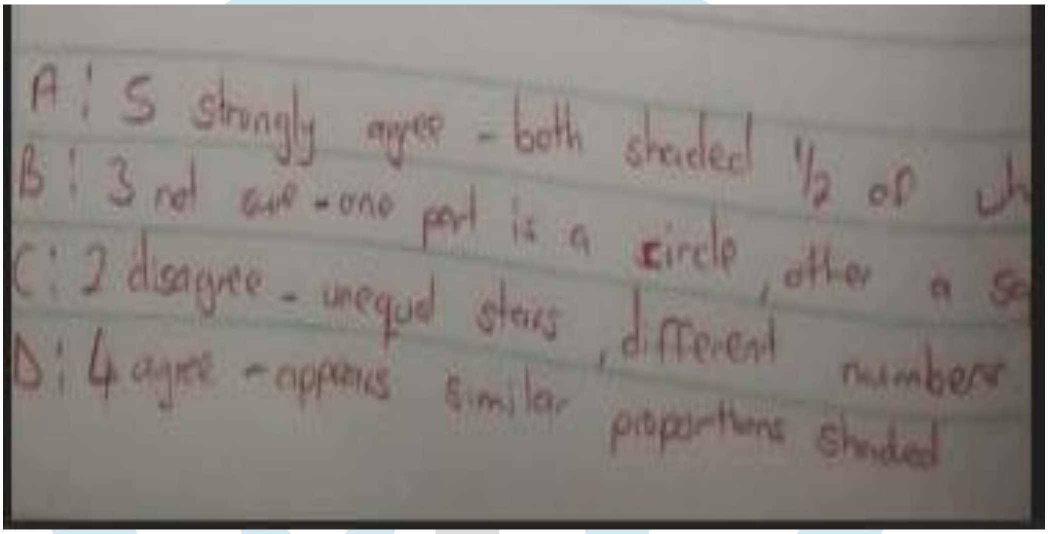


Fig. 4: Molly's Responses to items 6 A to D

Participant Molly was able to identify that item A represented a case of equivalence because each rectangle has a portion representing $\frac{1}{2}$. This is a case of quantitative variance and the participant indicated knowledge of it. However, quantitative invariance was observed in item B where the participant revealed that she doubted whether cases of different shapes could be compared. She indicated a '3', which said 'not sure'. She alleged that,

'*One part is a circle, the other a square*.

This could be an indication that the participant used the rectangle only in item 2 because she believed in structural invariance. This is similar to a pattern observed by [50] of a rigidity in the procedures exhibited by pre-service teachers in their knowledge of the concept image on fractions. Participant Molly held that one can only compare fractions represented with the same type of shape, that is, believed in structural invariance. In item 5 C, the participant reasoned that because the number of stars in the two sets in item 5 C was not equal, the two sets did not represent equivalent fractions (see Figure 4). This revealed that the participant was aware of unit equivalence and did not have background knowledge to understand proportional equivalence. Moreover, [47] alleges that to understand the equivalence of fractions effectively, one should be able to make connections among symbols, models, pictures, and context. The participant was able to identify that shapes in item 5 A represented equivalent fractions, yet in 5 C, could not observe that one set was $\frac{4}{12} = \frac{1}{3}$ and the other set was $\frac{6}{18} = \frac{1}{3}$. This indicated that the participant was not capable of handling the set format of representing fractions. She was also able to observe that item 5 D indicated equivalence. The inconsistencies revealed by this participant could be evidence that the participant could have been guessing some of the responses in the subsequent items.


A. Answer : 4 - agree.
- the first diagram shows a rectangle divided into 4 equal parts, with 2 parts shaded. This represents the fraction 2/4. The second diagram shows a rectangle, divided into 2 equal parts, with 1 part shaded. This represents the fraction 1/2. Since 2/4 is equivalent to 1/2, the diagrams show equivalence.

B. Answer : I strongly disagree.
- The first diagram shows a circle divided into 4 equal parts, with 1 part shaded. This represents the fraction 1/4. The second diagram shows a square divided into 4 parts, with 2 parts shaded. However, the parts are not equal, so it does not represent a fraction in the same way. The shaded area are not equivalent.

C. Answer 4 Agree.
- the first diagram shows 6 stars, with 3 of them shaded. This represents the fraction 3/6. The second diagram shows 4 stars, with 2 of them shaded. This represents the fraction 2/4. Both fractions simplify to 1/2, so the diagrams show equivalence.

D Answer : 4 Agree.
- The first diagram shows a rectangle divided into 3 equal parts, with 2 parts shaded. This represents the fraction 2/3. The second diagram shows a rectangle divided into 6 equal parts, with 4 parts shaded. This represents the fraction 4/6, since 4/6 is equal to 2/3, the diagram shows equivalence.

Fig. 5: Diva's responses to items 6 A to D

Participant Diva indicated that the shapes in item 5 A showed equivalent fractions. All shapes were rectangles, and the fractions were represented using the area model. The participant's explanation of the choice clearly indicated specialised knowledge of equivalent fractions (see Figure 5). Like Molly, participant Diva alleged that item 5 B included a circle and a rectangle and therefore disagreed that they could indicate equivalence. Participant Diva wrote,

> *'I strongly disagree. The first diagram shows a circle divided into 4 equal parts with one part shaded. This represents the fraction $\frac{1}{4}$. The second diagram shows a square divided into 4 parts with 2 parts shaded. However, the parts are not equal, so it does not represent a fraction in the same way. The shaded areas are not equivalent.*

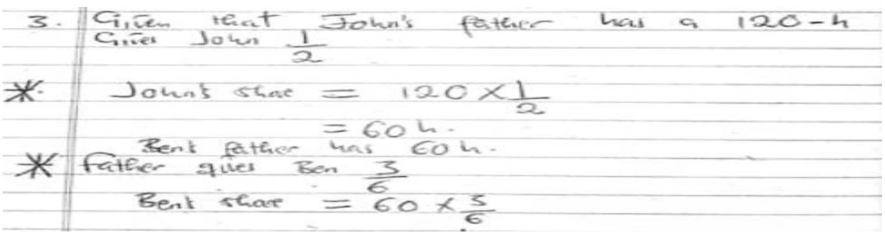
This is a pre-service teacher who has finished his one year of teaching practice. He did not understand the extent to which fraction equivalence could go despite having gone through the 3-3-3 teacher education for two full years, one year theory and the other year, practical. He continued to carry misconceptions [33] and pass them on to learners who would become teachers in the future if they chose to be.

Research Question 3: To what extent is the 3-3-3 elementary teacher education model

effective in equipping pre-service teachers with the knowledge of different forms of fraction equivalency?

Items 3 and 4 answered research question 3.

Item 3: John's father has a field which is 120 hectares, and gave him $\frac{1}{2}$ of the field. Ben's father has 60 hectares and gave him $\frac{3}{6}$ of the field. How many hectares does each get? Is John's share equivalent to Ben's share? Give a reason.


3. Given that John's father has a 120-h
Gives John 1/2
* John's share = 120 × 1/2
= 60 h.
Ben's father has 60 h.
* Father gives Ben 3/6
Ben's share = 60 × 3/6

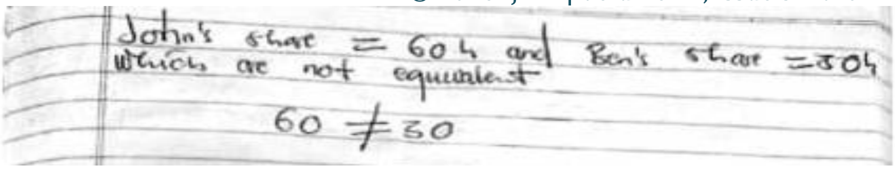
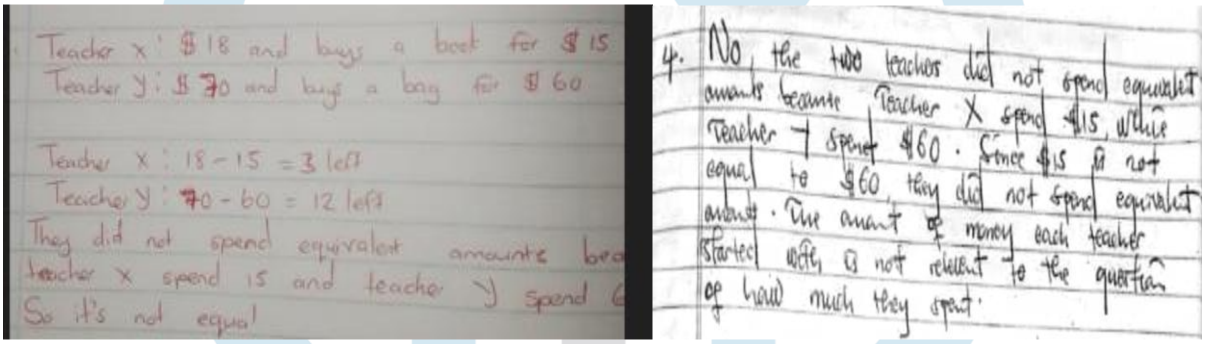


Fig. 6: Diva's response to item 3

Participant Diva got both calculations correct. However, the participant went on to conclude that because

60 ≠ 30, the two shares were not equivalent. Participant Diva did not regard the fact that $\frac{1}{2}$ is equivalent to $\frac{3}{6}$ and only compared John's 60 hectares to Ben's 30 hectares, which are clearly not equal without any reference to the origins of the quantities (see Figure 6). This reveals a limited understanding of equivalence of fractions and is consistent with [50] assertion that pre-service teachers in their study's understanding of the concepts of fraction comparison are grounded in standardised procedures. The participant in this study expected some shapes to be drawn and shaded for her to observe equivalence. This also indicated that participant Diva did not have knowledge of proportional equivalence.

Item 4: Teacher X has $18 and buys a book for $15, and Teacher Y has $72 and buys a bag for $60. Have the two teachers spent equivalent amounts? Give a reason?



Molly Diva

Fig. 7: Responses to item 4

Molly subtracted the amounts spent from the original amounts and then commented that the two teachers did not spend equivalent amounts because teacher X spent 15 and teacher Y spent 60. This response shows that pre-service teacher participant Molly had very limited to no specialised knowledge of proportional equivalence of quantities. Instead of changing the numbers to fractions, the participant chose to subtract, indicating a limited knowledge of the purpose of fractions and equivalent fractions when comparing.

Participant Diva, also regarded the two amounts spent not equivalent saying because teacher X spent $15 and teacher Y spent $60 which are not equal (see Figure 7). Participant Diva further said the starting amounts were not relevant. The participant also failed to compare the amounts using fractions. The participant did not have knowledge of proportional equivalence where one should change the amounts to fractions and compare. The two participants revealed that they found equivalent fractions difficult. Moreover, [3] indicated in a study that meaningful learning and construction of equivalent fractions is identified as difficult for many learners and teachers. This revealed that the two participants had not been equipped well in teaching equivalent fractions in the 3-3-3 teacher education model.

CONCLUSION

The results of the study revealed that the two participants in the study's knowledge of equivalence were limited to the unit equivalence where the wholes are equal, that is, quantitative invariant. They also revealed that they had structural invariance as they both indicated they could not compare fractions using different shapes. Both participants further indicated limited knowledge of the use of fractions, supporting [12] findings that pre-service teachers also struggle with the concept of fractions. The participants indicated that the 3-3-3 teacher education model had not equipped them with enough knowledge on fraction equivalence to the extent that they would know both unit equivalence and proportional equivalence. Participants revealed that they had knowledge of the unit equivalence thereby limiting their teaching effectiveness. These findings were disturbing considering [12] statement that such a deficiency of deep and flexible knowledge base has devastating implications to the pre-service teachers' performance in class teaching and learning implying that teachers cannot teach things they themselves do not know effectively.

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DECLARATION OF INTEREST STATEMENT

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

I, Dambakushamba Shorayi, declare that this manuscript is original, has not been published before and is not currently being considered for publication elsewhere. All participants provided written consent for inclusion before they participated in the study.

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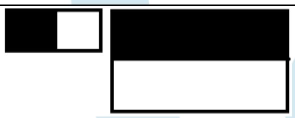
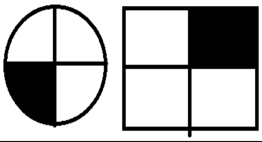
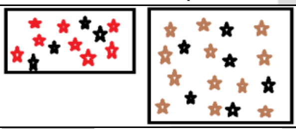
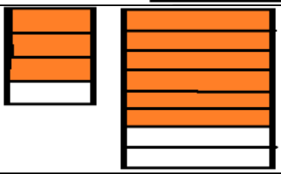
Appendix A

Can you please do the following exercise for me?

Fraction Equivalency Test

Attempt All Questions

1. Fractions are equivalent when-------------------------.
2. Draw diagrams to show that $\frac{2}{3} = \frac{4}{6}$.
3. John's father has a field which is 120 hectares, and gave him $\frac{1}{2}$ of the field. Ben's father has 60 hectares and gave him $\frac{3}{6}$ of the field. How many hectares does each get? Is John's share equivalent to Ben's share? Give a reason.
4. Teacher X has $18 and buys a book for $15, and Teacher Y has $72 and buys a bag for $60. Have the two teachers spend equivalent amounts? Give a reason?
5. The following diagrams show shaded parts. (Choose whether you **1-strongly disagree, 2-disagree, 3-not sure, 4-agree, 5-strongly agree** with the suggestion that the diagrams show equivalence**. Write item (e.g., A and the number of your choice, e.g. E-3) for your choice from the yellow strip.**

Item	Shape	1-Strongly disagree	2-Disagree	3-Not sure	4-Agree	5-Strongly agree
A						
B						
C						
D						

When through please take a picture of your responses and send me to my inbox.

Thank you for your time and effort.

Appendix B

From the Office of the Secretary for Higher and Tertiary Education, Innovation, Science and Technology Development

All official communications should be addressed to:

"The Secretary"

Telephones: 795891-5, 796441-9, 730055-9
Fax Number: 732070
Telegraphic address: "EDUCATION"

ZIMBABWE

Reference:

MINISTRY OF HIGHER AND TERTIARY EDUCATION, INNOVATION, SCIENCE AND TECHNOLOGY DEVELOPMENT

P. BAG CY 7732
CAUSEWAY

STAFF CONFIDENTIAL

21 November 2023

Dear Mr. S. Dambakushamba

APPROVAL TO CARRY OUT RESEARCH: MR SHORAYI DAMBAKUSHAMBA: UNIVERSITY OF SOUTH AFRICA

Reference is made to your application in which you requested for authority to carry out research at the [redacted] College and [redacted] Teachers College in the Ministry of Higher and Tertiary Education, Innovation, Science and Technology Development.

Please note that the Head of Ministry, has granted you authority to carry out your research entitled, **'Examining pre-service teachers' specialized mathematical 'knowledge for teaching fractions at Grade 6 level: An explanatory study of the 3-3-3 elementary Teacher Education Model.**

It is hoped that your research will benefit the Ministry and it would be appreciated if you could supply the office of the Permanent Secretary with a final copy of your study, as the findings would be relevant to the Ministry's strategic planning process.

MIN. OF HIGHER & TERTIARY EDU., INNOVATION, SCIENCE & TECHN. DEV.
HUMAN RESOURCES
24 NOV 2023
P. BAG 7732, CAUSEWAY
ZIMBABWE

E. Yesaya
Human Resources Officer
FOR: Secretary
Higher and Tertiary Education, Innovation, Science and Technology Development

C.c The Principal [redacted] Teachers College
The Principal [redacted] College
File

19 JAN 2024

Received

[redacted]